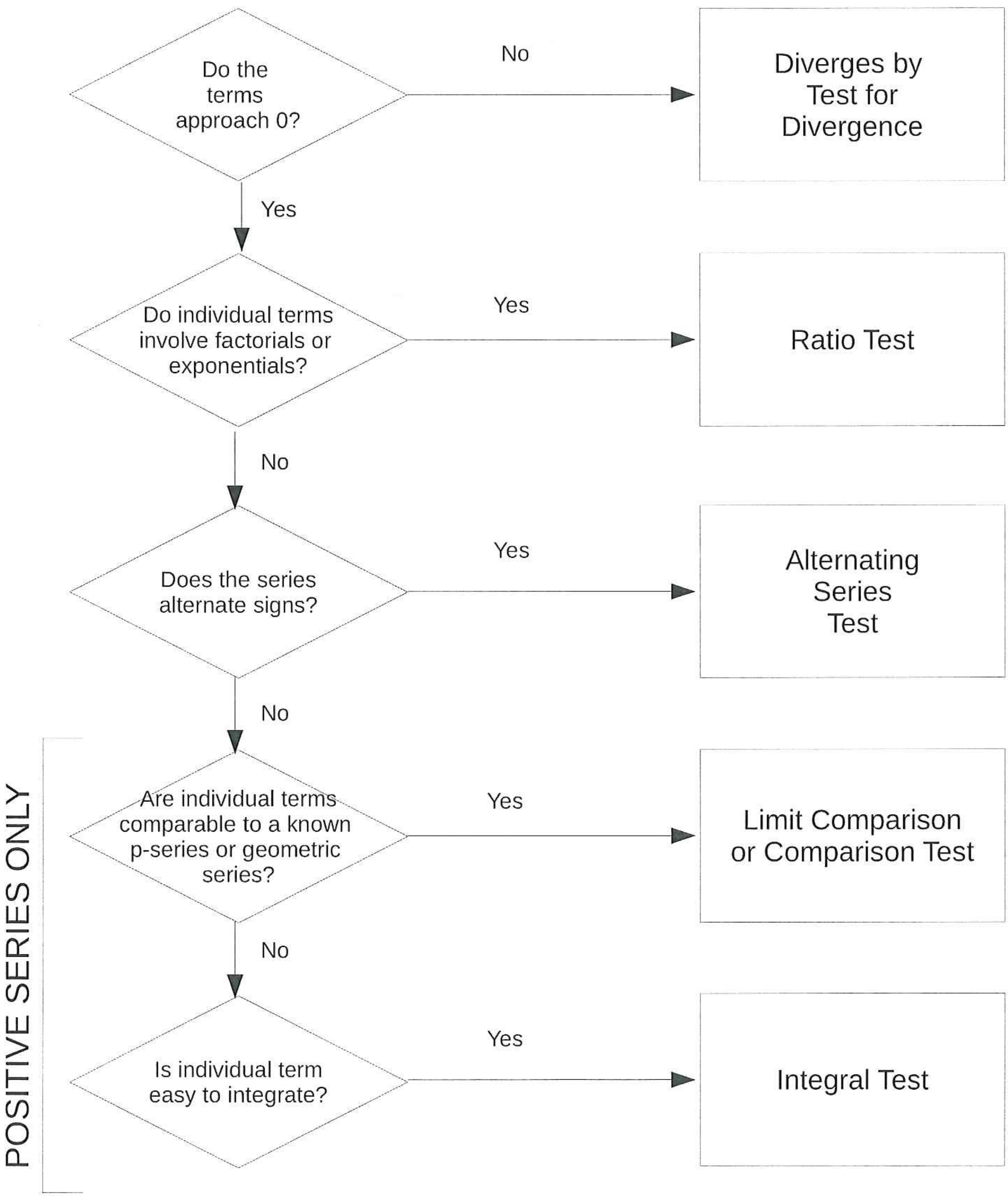
Choosing a Series Convergence Test



SERIES-TESTS FOR CONVERGENCE

1. **Geometric series**: The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. If $|r| < 1$, then the sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

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2. **The Integral Test**: If $f(x)$ is a positive, continuous, decreasing function on $[1, \infty]$, and $a_n = f(n)$. Then:

   a.) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ converges.

   b.) If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

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3. The **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

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4. **The Comparison Test**: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{i=1}^{\infty} b_n$ are series of positive terms.

   a.) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

   b.) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

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5. **The Limit Comparison Test**: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of positive terms.

   a.) If $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both diverge.

   b.) If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

   c.) If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

<u>When to use the limit comparison test</u>: When $\sum_{n=1}^{\infty} a_n$ is a series involving powers of n, look at the leading term in the numerator and denomonator and "ignore" all other powers of n. Simplify this expression. The result is $\sum_{n=1}^{\infty} b_n$. Use this series to apply the limit comparision test. [NOTE: $\sum_{n=1}^{\infty} b_n$ is almost always a p-series].

example 1: $\sum_{n=1}^{\infty} \frac{n^3+50n}{n^5+\sqrt{n}}$. Use $b_n = \frac{n^3}{n^5} = \frac{1}{n^2}$

example 2: $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+n^2+1}}$ Use $b_n = \frac{n}{\sqrt{n^5}} = \frac{1}{n^{3/2}}$

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6. **The Alternating Series Test:** If the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ satisfies

   a.) $a_{n+1} \leq a_n$ for all n (ie the sequence a_n is decreasing).

   b.) $\lim_{n\to\infty} a_n = 0$

   then the series converges.

When to use the alternating series test: When you have an alternating series!

7. **The Ratio Test:**

   a.) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

   b.) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

   c.) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test fails.

When to use the ratio test: Use the ratio test if the series involves $n!$ or numbers raised to the n^{th} power, such as 2^n or 5^{n+1}...NEVER USE THE RATIO TEST IF THE SERIES ONLY INVOLVES POWERS OF n. (Why??) Try it and see what happens. You should always get 1 as the limit, in which case the test fails. REMEMBER: If the series only involves powers of n and the series is a series of positive terms, use the limit comparision test.

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8. **The Test for Divergence** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. NOTE: The converse is NOT true: If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ does NOT NECESSARILY CONVERGE! For example the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, however the TERMS do go to zero-just not fast enough to get a finite SUM.

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9. **Remainder formulas:**

$$R_n \leq \int_n^{\infty} f(x)\,dx \qquad\qquad |R_n| \leq a_{n+1}$$