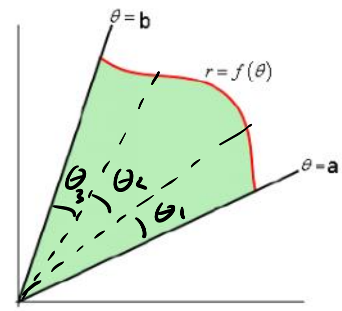
Section 10.4: Areas and Length in Polar Coordinates

We would like to find the area of the region that is between the pole (origin) and the polar equation $r = f(\theta)$ from $\theta = a$ to $\theta = b$.

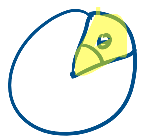


$r = f(\theta)$

To be able to find this area we start back with the area of a circle being $A = \pi r^2$.

A sector of a circle, which is a part of the circle formed by the central angle θ, has an area that is proportional to the whole circle.

$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}r^2\theta$



$\frac{1}{2} r_i^2 \Delta\theta$

$\frac{1}{2}(f(\theta))^2 \Delta\theta$

Now partition the region(on the right) where $\theta_1 = a$ to $\theta_n = b$. The area of each of the smaller sectors is given by $A_i = \frac{1}{2}r_i^2\Delta\theta$. Then area of the region is approximated by $A \approx \sum \frac{1}{2}r_i^2\Delta\theta$.

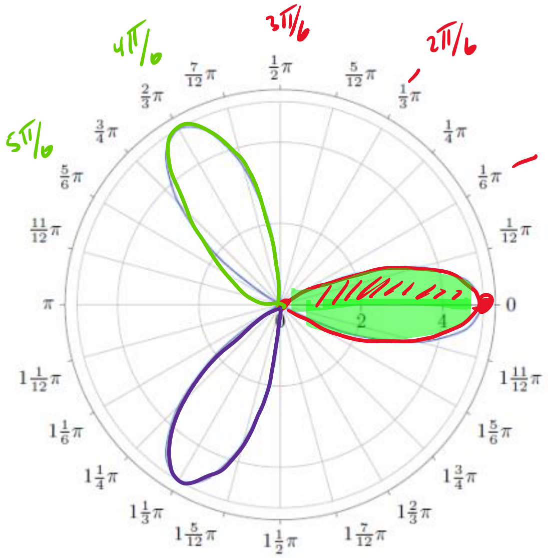
Thus the area of the region is $A = \displaystyle\int_a^b \frac{1}{2}r^2 d\theta$, where $r = f(\theta)$.

$$\lim \sum \frac{1}{2} r_i^2 \Delta\theta = \int_a^b \frac{1}{2} r^2 d\theta = \int_a^b \frac{1}{2}(f(\theta))^2 d\theta = \text{Area}$$

Pg 2

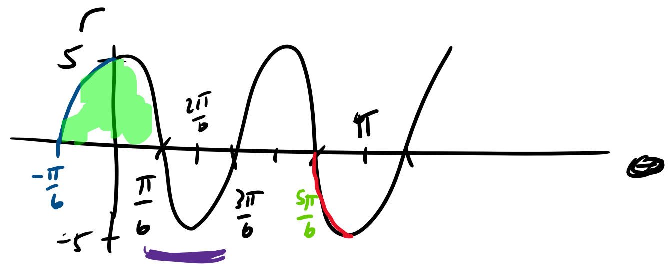
Wednesday, November 20, 2019 4:14 PM

Example: Find the area of one petal of the graph $r = 5\cos(3\theta)$.



$3\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{6}$



$$A = \int_{-\pi/6}^{\pi/6} \frac{1}{2}\left(5\cos(3\theta)\right)^2 \, d\theta$$

$$A = 2\int_0^{\pi/6} \frac{1}{2}\left(5\cos(3\theta)\right)^2 \, d\theta$$

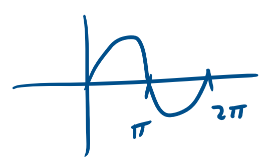
$$= \int_0^{\pi/6} 25\cos^2(3\theta)\, d\theta = \int_0^{\pi/6} 25 \cdot \frac{1}{2}\left[1 + \cos(6\theta)\right] d\theta$$

$$= \frac{25}{2}\int_0^{\pi/6} 1 + \cos(6\theta)\, d\theta = \frac{25}{2}\cdot\left[\theta + \frac{1}{6}\sin(6\theta)\right]_0^{\pi/6}$$

$$= \frac{25}{2}\left[\frac{\pi}{6} + \frac{1}{6}\sin(\pi) - \left(0 + \frac{1}{6}\sin(0)\right)\right]$$

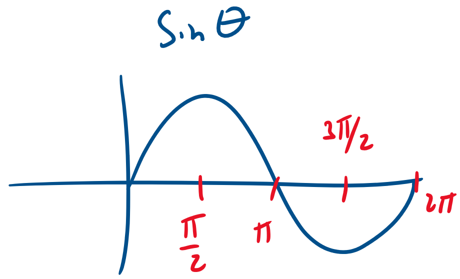
$$= \frac{25}{2}\left[\frac{\pi}{6} + 0\right]$$

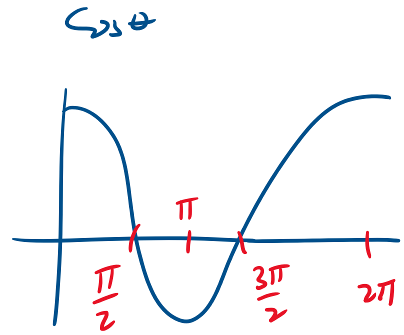
$$= \frac{25\pi}{12}$$



$\sin\theta$

$\cos\theta$





$\cos(\underbrace{3\theta})$

$3\theta = \frac{\pi}{2}$

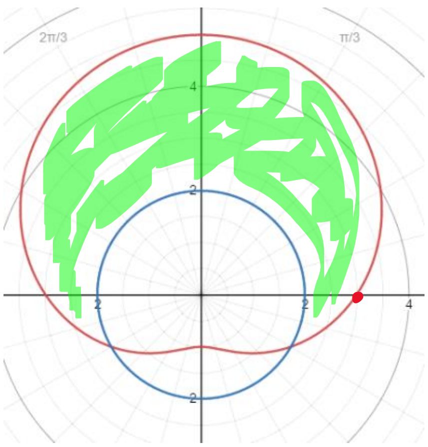
$\theta = \frac{\pi}{6}$

$$\text{Area} = \int_a^b \frac{1}{2} r^2 \, d\theta$$

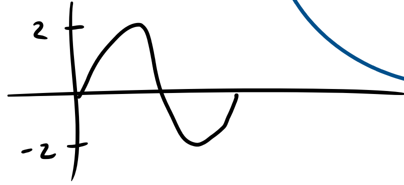
Example: Find the area inside $r = 3 + 2\sin\theta$ and outside the circle $r = 2$.

$$\text{Area} = \int_a^b \frac{1}{2} r_o^2 - \frac{1}{2} r_i^2 \, d\theta$$

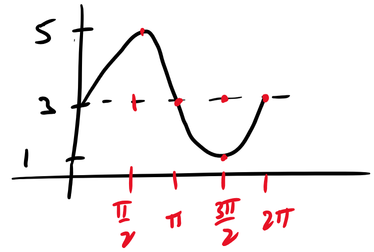
outer Inner



$2\sin\theta$



$3 + 2\sin\theta$



$3 + 2\sin\theta = 2$

$2\sin\theta = -1$

$\sin\theta = -\frac{1}{2}$

$\theta = -\frac{\pi}{6}, \frac{7\pi}{6}$

$$\text{Area} = \int_{-\pi/6}^{7\pi/6} \frac{1}{2}\left[(3+2\sin\theta)^2 - (2)^2\right] d\theta$$

$$= 2\int_{-\pi/6}^{\pi/2} \frac{1}{2}\left[(3+2\sin\theta)^2 - 4\right] d\theta$$

$$= \int_{-\pi/6}^{\pi/2} 9 + 12\sin\theta + 4\sin^2\theta - 4 \, d\theta = \int_{-\pi/6}^{\pi/2} 5 + 12\sin\theta + 4\sin^2\theta \, d\theta$$

$$= \int_{-\pi/6}^{\pi/2} 5 + 12\sin\theta + 4\cdot\frac{1}{2}\left[1 - \cos 2\theta\right] d\theta$$

$$= \left(5\theta - 12\cos\theta + 2\left[\theta - \frac{1}{2}\sin 2\theta\right]\right)\Bigg|_{-\pi/6}^{\pi/2}$$

$$\left(5\theta - 12\cos\theta + 4\left[\theta - \frac{1}{2}\sin 2\theta\right]\right)\Bigg|_{-\pi/6}$$

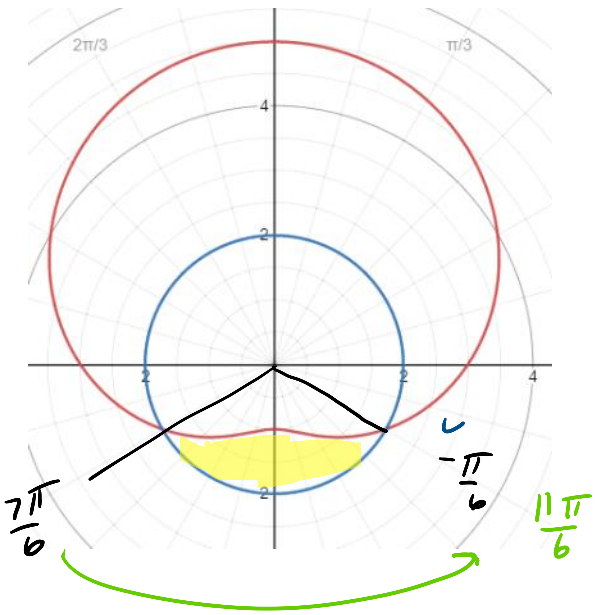
$$= \left(5\theta - 12\cos\theta + 2\theta - \sin 2\theta\right)\Bigg|_{-\pi/6}^{\pi/2}$$

$$= \left(7\theta - 12\cos\theta - \sin 2\theta\right)\Bigg|_{-\pi/6}^{\pi/2} = \cdots = \frac{11\sqrt{3}}{2} + \frac{14\pi}{3}$$

Pg 4

Wednesday, November 20, 2019 4:14 PM

Example: Find the area inside the circle $r = 2$ and outside $r = 3 + 2\sin\theta$



$$\text{Area} = \int_{7\pi/6}^{11\pi/6} \frac{1}{2}\left[(2)^2 - (3+2\sin\theta)^2\right] d\theta$$

$$= 2\int_{7\pi/6}^{3\pi/2} \frac{1}{2}\left[4 - (3+2\sin\theta)^2\right] d\theta$$

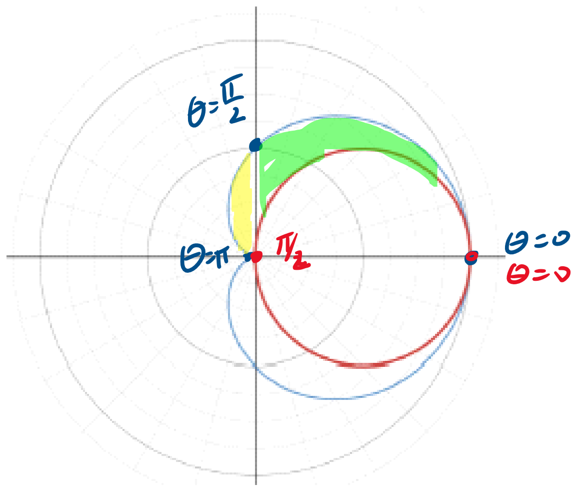
$$= \ldots = \frac{11\sqrt{3}}{2} - \frac{2\pi}{3}$$

Pg 5

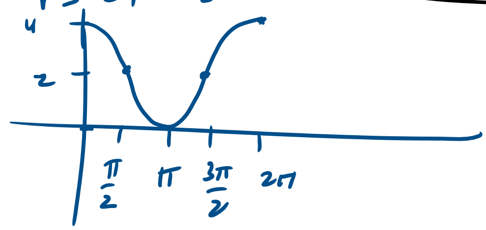
Wednesday, November 20, 2019 4:14 PM

Example: Setup the integral(s) that give the area above the x-axis and inside $r = 2 + 2\cos\theta$ and outside $r = 4\cos\theta$

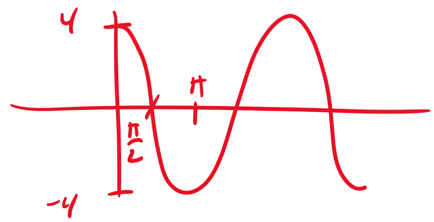
$$\boxed{\begin{aligned} r &= 4\cos\theta \\ r^2 &= 4r\cos\theta \\ x^2+y^2 &= 4x \end{aligned}}$$



$r = 2+2\cos\theta$



$r = 4\cos\theta$



$$\underbrace{\int_0^{\pi/2} \frac{1}{2}\left[(2+2\cos\theta)^2 - (4\cos\theta)^2\right] d\theta}_{\text{outside} \qquad \text{Inside.}} + \underbrace{\int_{\pi/2}^{\pi} \frac{1}{2}(2+2\cos\theta)^2 \, d\theta}$$

Pg 6 arc length

Wednesday, November 20, 2019 4:14 PM

Arc Length

From section 10.2 we know the length of a curve is $L = \int_a^b ds$ where $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.$

Find the arc length of the polar curve $r =$ for $a \le \theta \le b$. Once again we assume that the curve is traced exactly once.

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

We start with $x = r\cos\theta$ and $y = r\sin\theta$ or $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$
We know the formula for ds.

$r = f(\theta)$

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta = \cdots \text{lots of algebra} \cdots = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$\text{arc length} = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta = \int_a^b \sqrt{(f(\theta))^2 + \left(\frac{df}{d\theta}\right)^2}\ d\theta$$

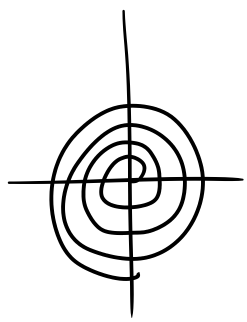
Example: Find the length of the curve $r = \theta$ for $0 \le \theta \le 1$.

$$\frac{dr}{d\theta} = r' = 1$$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^1 \sqrt{\theta^2 + (1)^2}\, d\theta$$

$$= \int_0^1 \sqrt{\theta^2 + 1}\, d\theta \rightarrow \int_0^1 \sqrt{x^2+1}\, dx$$

$r = \theta \qquad \theta > 0$



$x = \tan\theta$ $\qquad x=0 \rightarrow \theta = 0$

$dx = \sec^2\theta\, d\theta$ $\qquad x=1 \rightarrow \theta = \frac{\pi}{4}$

$$\int_{x=0}^{x=1} \sqrt{\tan^2\theta + 1} \cdot \sec^2\theta\, d\theta$$

$$= \int_{x=0}^{x=1} \sqrt{\sec^2\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int_{x=0}^{x=1} \sec^3\theta\, d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| \Bigg]_{\theta=0}^{\theta=\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right)$$

Pg 8

Wednesday, November 20, 2019 4:14 PM

Example: Find the length of the curve $r = -4\sin\theta$ for $0 \le \theta \le \frac{2\pi}{3}$

$$L = \int_a^b \sqrt{r^2 + (r')^2}\, d\theta$$

$r' = -4\cos\theta$

$$L = \int_0^{2\pi/3} \sqrt{(-4\sin\theta)^2 + (-4\cos\theta)^2}\, d\theta = \int_0^{2\pi/3} \sqrt{16\sin^2\theta + 16\cos^2\theta}\, d\theta$$

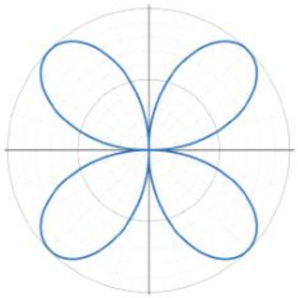
$$= \int_0^{2\pi/3} \sqrt{16}\, d\theta = \int_0^{2\pi/3} 4\, d\theta = 4\theta \Big|_0^{2\pi/3} = 4\left(\frac{2\pi}{3}\right)$$

$$= \frac{8\pi}{3}$$

Pg 9

Wednesday, November 20, 2019 4:14 PM

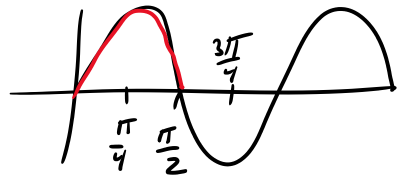
Example: Setup the integral that would give the length of the curve that forms one of the loops for $r = \sin(2\theta)$.



$r' = 2\cos(2\theta)$

$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$



$$\int_0^{\pi/2} \sqrt{(\sin(2\theta))^2 + (2\cos(2\theta))^2}\, d\theta$$