Application of Taylor Polynomials

Taylor Polynomials.

The Taylor series of a function, $f(x)$, can be expressed: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

The **n-th degree Taylor polynomial** of $f(x)$ at a, denoted T_n is given by

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

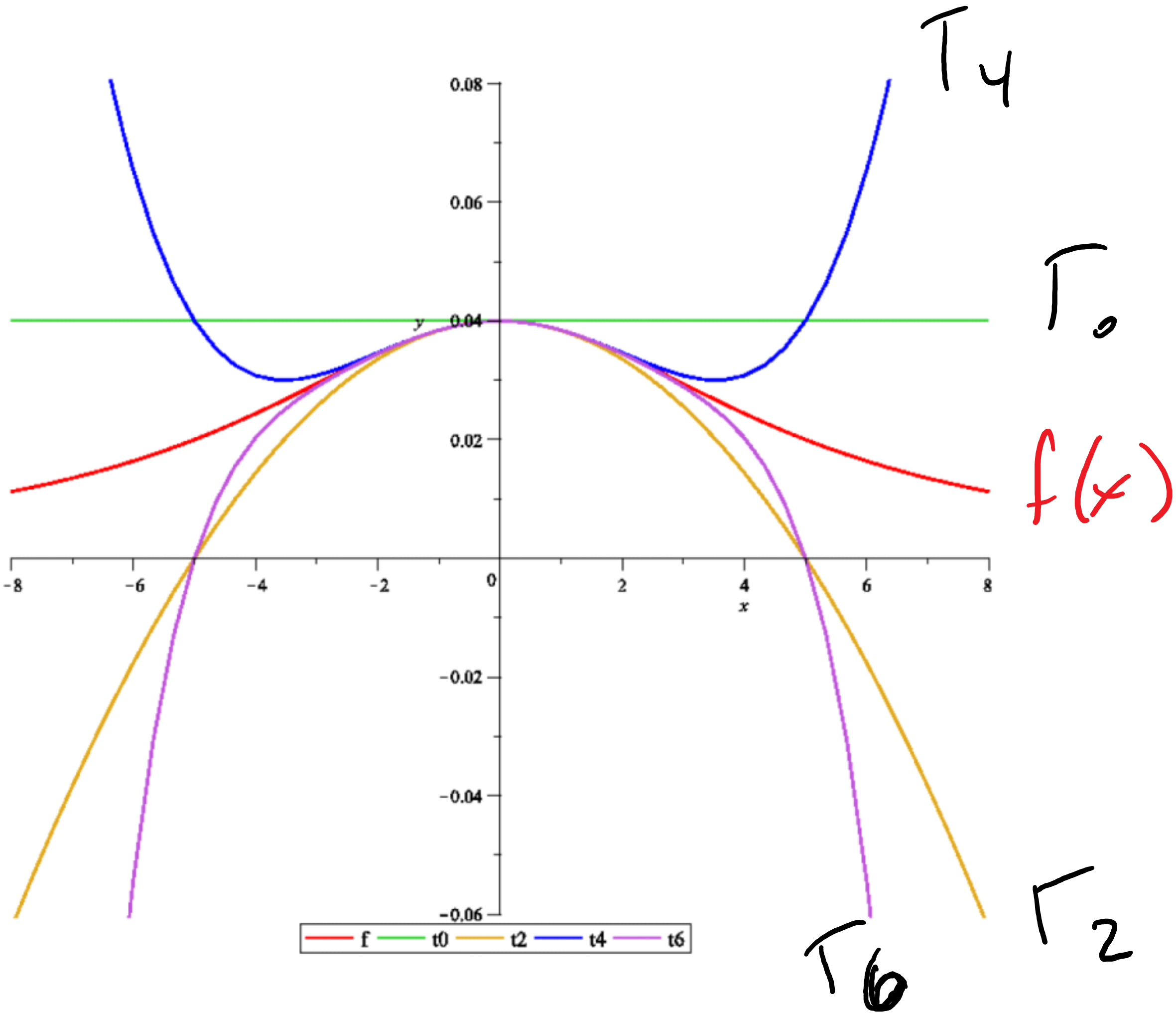
degree n i.e. $(x-a)^n$

$f(x)$ @ $x=a$

Tangent line

$y = f(a) + f'(a)(x-a)$

The following graph shows the function $f(x) = \dfrac{10}{x^2 + 25}$ and T_0, T_2, T_4, and T_6.



Example: Find the Taylor polynomials, T_1, T_2, and T_3, for $f(x) = xe^x$ centered at $a = 2$.

$$T_3 = \underbrace{\overbrace{c_0 + c_1(x-2)}^{T_1} + c_2(x-2)^2}_{T_2} + c_3(x-2)^3$$

$f(x) = xe^x$ $\quad$ $f(2) = 2e^2$ $\quad$ $c_0 = \frac{f(2)}{0!}$

$f' = = 1e^x + x \cdot e^x = (1+x)e^x$ $\quad$ $f'(2) = 3e^2$ $\quad$ $c_1 = \frac{f'(2)}{1!}$

$f'' = 1e^x + (1+x)e^x = (2+x)e^x$ $\quad$ $f''(2) = 4e^2$ $\quad$ $c_2 = \frac{f''(2)}{2!}$

$f''' = 1e^x + (2+x)e^x = (3+x)e^x$ $\quad$ $f'''(2) = 5e^2$ $\quad$ $c_3 = \frac{f'''(2)}{3!}$

$$T_1 = c_0 + c_1(x-2) = 2e^2 + 3e^2(x-2)$$

$$T_2 = c_0 + c_1(x-2) + c_2(x-2)^2 = 2e^2 + 3e^2(x-2) + \frac{4e^2}{2}(x-2)^2$$

$$T_3 = c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 = 2e^2 + 3e^2(x-2) + \frac{4e^2}{2}(x-2)^2 + \frac{5e^2}{3!}(x-2)^3$$

Example: Find the Taylor polynomials, T_1, T_4, T_5, and T_7 for $f(x) = \dfrac{x}{1+5x^3}$ centered at $a = 0$

$$\frac{x}{1+5x^3} = x \cdot \frac{1}{1-(-5x^3)} = x \cdot \sum_{n=0}^{\infty} (-5x^3)^n = x \sum_{n=0} (-1)^n 5^n x^{3n}$$

$$\frac{x}{1+5x^3} = \sum_{n=0}^{\infty} (-1)^n 5^n x^{3n+1}$$

$$= x - 5x^4 + 5^2 x^7 - 5^3 x^{10} + \cdots$$

$$= \underbrace{\underbrace{\underbrace{\underbrace{0 + x}_{} + 0x^2 + 0x^3 - 5x^4}_{} + 0x^5}_{} + 0x^6 + 5^2x^7}_{} + 0x^8 + 0x^9 - 5^3x^{10} + \cdots$$

$$T_1 = 0 + x = x$$

$$T_4 = 0 + x + 0x^2 + 0x^3 - 5x^4 = x - 5x^4$$

$$T_5 = 0 + x + 0x^2 + 0x^3 - 5x^4 + 0x^5 = x - 5x^4$$

$$T_7 = x - 5x^4 + 5^2 x^7$$

Example: Express $f(x) = 2x^3 + 4x^2 + 7x + 6$ as a Taylor polynomial(series) about $a = 2$.

$f(x) = 2x^3 + 4x^2 + 7x + 6$ $\quad f(2) = 52$

$f'(x) = 6x^2 + 8x + 7$ $\quad f'(2) = 47$

$f''(x) = 12x + 8$ $\quad f''(2) = 32$

$f'''(x) = 12$ $\quad f'''(2) = 12$

$f^{(4)}(x) = 0$

$a = 0$

$f(0) = 6$

$f'(0) = 7$

$f''(0) = 8$

$f'''(0) = 12$

$6 + 7(x-0) + \frac{8}{2!}(x-0)^2 + \frac{12}{3!}(x-0)^3$

$6 + 7x + 4x^2 + 2x^3$

$$T_3 = 52 + 47(x-2) + \frac{32}{2!}(x-2)^2 + \frac{12}{3!}(x-2)^3$$