Definition: Given a sequence $\{a_i\}$, we can construct an **infinite series** or **series** by adding the terms of the sequence. $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots$

Definition: The **nth partial sum** of a series, denoted s_n, is the sum of the first n-terms.

NOTE: If the index starts at $i = 1$ then

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

$$s_1 = a_1$$
$$s_2 = s_1 + a_2 = a_1 + a_2$$
$$s_3 = s_2 + a_3 = a_1 + a_2 + a_3$$
$$s_4 = s_3 + a_4 = a_1 + a_2 + a_3 + a_4$$
$$s_5 = s_4 + a_5 = a_1 + a_2 + a_3 + a_4 + a_5$$

$$S_n = S_{n-1} + a_n$$

$$S_n - S_{n-1} = a_n$$

Example: Find the s_4 for the series: $\sum_{i=4}^{\infty} \frac{1}{(i-2)^2} = \underbrace{\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}}_{S_4} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots$

$$S_4 = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$$

How To Shift a Series:

Example: Adjust the series $\sum_{i=3}^{\infty} 10\left(\frac{1}{3}\right)^{2i}$ so that the index will now start at i=1.

$j = i - 2$

$j + 2 = i$

$$\sum_{j=1}^{\infty} 10\left(\frac{1}{3}\right)^{2(j+2)} = \sum_{j=1}^{\infty} 10\left(\frac{1}{3}\right)^{2j+4}$$

$$\sum_{i=1}^{\infty} 10\left(\frac{1}{3}\right)^{2i+4}$$

Definition: Let $\sum_{i=1}^{\infty} a_i$ be a series with s_n being the nth partial sum of this series. If the **sequence of partial sums** $\{s_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$, then we say that the series $\sum_{i=1}^{\infty} a_i$ converges to s or that the series has a sum of s, $\sum_{i=1}^{\infty} a_i = s$. If $\{s_n\}$ does not converge, then the series $\sum_{i=1}^{\infty} a_i$ is said to be divergent.

$\sum a_i$

series — terms added together

partial sums

n	a_n
1	40
2	8
3	8/5 = 1.6
4	8/25 = 0.32
5	8/125 = 0.064
6	8/625 = 0.0128
7	8/ 3125 = 0.00256
8	8/15625 = 0.000512
9	8/78125 = 0.0001024
10	8/390625 = 0.00002048

n	s_n
1	40
2	48
3	49.6
4	49.92
5	49.984
6	49.9968
7	49.99936
8	49.999872
9	49.9999744
10	49.99999488

as $n \to \infty$ $\quad s_n \to 50$

Thus $\sum_{i=1}^{\infty} a_i = 50$

This series converges

Theorem: If the series $\sum_{i=1}^{\infty} a_i$ is convergent, then $\lim_{i\to\infty} a_i = 0$

Test for Divergence: If $\lim_{i\to\infty} a_i \neq 0$ or DNE, then the series $\sum_{i=1}^{\infty} a_i$ is divergent.

Example: Which of these series DO NOT have a chance at being convergent?

A) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$a_n = \frac{1}{n^3}$ as $n \to \infty$ $a_n \to 0$

may or may not converge.

B) $\sum_{n=1}^{\infty} \frac{3n+5}{7-2n}$

$a_n = \frac{3n+5}{7-2n}$ as $n \to \infty$ $a_n \to -\frac{3}{2}$

This series will diverge

C) $\sum_{n=1}^{\infty} \cos(e^{-n})$

$a_n = \cos(e^{-n})$ as $n \to \infty$ $\cos(e^{-n}) \to \cos(0) = 1$

This series will diverge.

Example: The series $\sum_{i=1}^{\infty} a_i$ has a nth partial sum given by s_n. Will the series converge or diverge? Find the formula for the a_n term.

$$s_n = \frac{3n+5}{7-2n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3n+5}{7-2n} = \frac{3}{-2} = -1.5$$

partial sum

serie $\sum_{i=1}^{\infty} a_i$ will converge to $\underline{\underline{-1.5}}$

$$a_n = S_n - S_{n-1}$$

$$\boxed{a_n = \frac{3n+5}{7-2n} - \frac{3(n-1)+5}{7-2(n-1)}}$$

find a_4

$$a_4 = S_4 - S_3$$
$$= \frac{17}{-1} - \frac{14}{1}$$
$$= -17 - 14 = -31$$

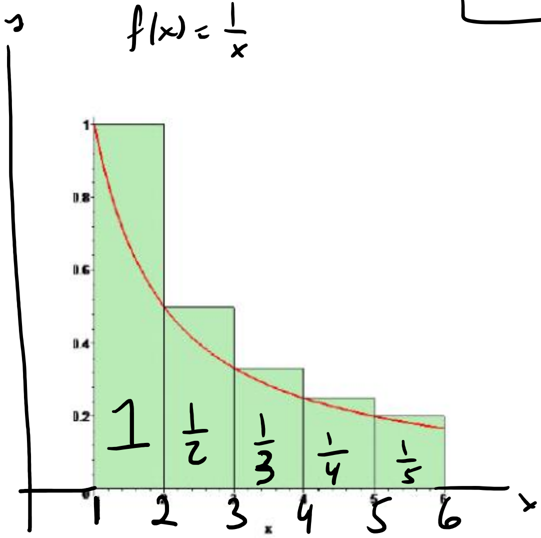
$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots$

Example: Determine if the Harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, converges or diverges.

$a_n = \frac{1}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$f(x) = \frac{1}{x}$



$$\underbrace{\int_1^{\infty} \frac{1}{x}\,dx}_{} < \sum_{n=1}^{\infty} \frac{1}{n}$$

p-integral
$p=1$
diverges.

by comparison
$\sum_{n=1}^{\infty} \frac{1}{n}$ will div.

Example: The **geometric series** may be defined in a variety of methods.

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ...$$

$$\sum_{n=0}^{\infty} ar^{n} = a + ar + ar^2 + ar^3 + ...$$

$$\sum_{n=7}^{\infty} ar^{n-7} = a + ar + ar^2 + ar^3 + ...$$

if $|r|<1$ conv.

Sum is $\frac{a}{1-r}$

Proof of the Geometric Series:

Consider the parital sum of the first n terms.

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

Multiply S_n by r to get: $rS_n = ar + ar^2 + ar^3 + ... + ar^n$

Now compute $S_n - rS_n$ and then solve for S_n

$$S_n - rS_n = a - ar^n$$

$$(1-r)S_n = a - ar^n$$

$$S_n = \frac{a - ar^n}{1-r}$$

$$\text{Sum} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1-r} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases}$$

Theorem: If $\sum a_n$ and $\sum b_n$ are covergent series, then so are the following series

$\sum ca_n = c \sum a_n$ (where c is a constant)

$\sum (a_n + b_n) = \sum a_n + \sum b_n$

$\sum (a_n - b_n) = \sum a_n - \sum b_n$

Example: Determine if these series are convergent or divergent. If the series is convergent, then give the sum of the series.

A) $1 - \frac{4}{3} + \frac{16}{9} - \frac{64}{27} + ...$

$-\frac{4}{3}$ $-\frac{4}{3}$ $-\frac{4}{3}$

geometric series.

$a = 1$ $r = -\frac{4}{3}$

Since $|r| = \left|-\frac{4}{3}\right| = \frac{4}{3}$ is not <1 The series will diverge.

$$\underbrace{a}\ +\ \underbrace{ar}\ +ar^2\ +ar^3\ +\cdots$$

B) $\sum_{\underline{\underline{i=3}}}^{\infty} 10\left(\frac{1}{3}\right)^{i-1} = \underbrace{10\left(\frac{1}{3}\right)^2} + \underbrace{10\left(\frac{1}{3}\right)^3} + 10\left(\frac{1}{3}\right)^4 + 10\left(\frac{1}{3}\right)^5 + \cdots$

$r = \frac{1}{3}$ $\qquad a = 10\left(\frac{1}{3}\right)^2 = \frac{10}{9}$ $\qquad \frac{ar}{a} = r$

$|r| < 1$

<u>Conv.</u>

$$\text{Sum} = \frac{a}{1-r} = \frac{\frac{10}{9}}{1-\frac{1}{3}} = \frac{\frac{10}{9}}{\frac{2}{3}} = \frac{10}{9}\cdot\frac{3}{2} = \frac{30}{18}$$

$$= \frac{15}{9} = \frac{5}{3}$$

C) $\sum_{n=0}^{\infty} 7 * 4^{-n} 3^{n-1} = \sum_{n=0} 7 \cdot \frac{3^{n-1}}{4^n}$

$= 7 \cdot \frac{3^{-1}}{4^0} + \frac{7 \cdot 3^0}{4^1} + 7 \cdot \frac{3^1}{4^2} + 7 \cdot \frac{3^2}{4^3} + 7 \cdot \frac{3^3}{4^4} + 7 \frac{3^4}{4^5} + \dots$

$n=0$ $n=1$ $n=2$ $n=3$

$r = \frac{3}{4}$

Since $|r| < 1$

Series will conv.

$a = \frac{7 \cdot 3^{-1}}{4^0} = \frac{7}{3}$

$\text{Sum} = \frac{a}{1-r} = \frac{7/3}{1 - 3/4}$

$= \frac{\frac{7}{3}}{\frac{1}{4}} = \frac{7}{3} \cdot \frac{4}{1} = \frac{28}{3}$

not geometric

D) $\sum_{i=1}^{\infty} \ln\left(\frac{i}{i+1}\right) = \sum_{i=1}^{\infty} \left[\ln(i) - \ln(i+1)\right]$

Test for div.

$\lim_{i\to\infty} \ln\left(\frac{i}{i+1}\right) = \ln(1) = 0$

may or may not conv.

Lets look at the partial sum.

$S_n = \ln(1) - \ln(2)$ $\quad i=1$

$+ \ln(2) - \ln(3)$ $\quad i=2$

$+ \ln(3) - \ln(4)$ $\quad i=3$

$+ \ln(4) - \ln(5)$ $\quad i=4$

$\vdots$

$i=n-3 \quad + \ln(n-3) - \ln(n-2)$

$i=n-2 \quad + \ln(n-2) - \ln(n-1)$

$i=n-1 \quad + \ln(n-1) - \ln(n)$

$i=n \quad + \ln(n) - \ln(n+1)$

$S_n = \ln(1) - \ln(n+1) \leftarrow$ n^{th} partial sum

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(1) - \ln(n+1) = -\infty$ Thus The Series diverges

E) $\sum_{i=3}^{\infty} \left(\frac{1}{i-2} - \frac{1}{i} \right)$

partial sum. $= 1 - \frac{1}{3}$ $\quad i=3$

$+ \frac{1}{2} - \frac{1}{4}$ $\quad i=4$

$+ \frac{1}{3} - \frac{1}{5}$ $\quad i=5$

$+ \frac{1}{4} - \frac{1}{6}$ $\quad i=6$

$+ \frac{1}{5} - \frac{1}{7}$ $\quad i=7$

$\vdots$

$i=n-3 \quad + \frac{1}{n-5} - \frac{1}{n-3}$

$i=n-2 \quad + \frac{1}{n-4} - \frac{1}{n-2}$

$i=n-1 \quad + \frac{1}{n-3} - \frac{1}{n-1}$

$i=n \quad + \frac{1}{n-2} - \frac{1}{n}$

partial sum $= 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$

$$\lim_{n\to\infty} \text{partial sum} = \lim_{n\to\infty} 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n} = 1 + \frac{1}{2} = \frac{3}{2}$$

The series converges to $3/2$

F) $\displaystyle\sum_{i=1}^{\infty} e^{5/(i+1)} - e^{5/i}$

$$
\begin{aligned}
S_n &= \cancel{e^{5/2}} - e^{5} && i=1\\
&+ \cancel{e^{5/3}} - \cancel{e^{5/2}} && i=2\\
&+ \cancel{e^{5/4}} - \cancel{e^{5/3}} && i=3\\
&\ \vdots\\
&\ \vdots\\
&+ \cancel{e^{5/n-1}} - \cancel{e^{5/n-2}} && i=n-2\\
&+ \cancel{e^{5/n}} - \cancel{e^{5/n-1}} && i=n-1\\
&+ e^{5/n+1} - \cancel{e^{5/n}} && i=n
\end{aligned}
$$

$$S_n = -e^{5} + e^{5/n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -e^{5} + e^{5/n+1} = -e^{5} + e^{0} = 1 - e^{5}$$

The series converges to $1-e^{5}$

Example: Use a geometric series to express $0.\overline{14}$ as a ratio of integers.

$$.\overline{14} = .14\,14\,14\,14\,14\ldots$$

$$= .14 + .0014 + .000014 + .00000014 + \ldots$$

$$= \frac{14}{100} + \frac{14}{100}\cdot\frac{1}{100} + \frac{14}{100}\cdot\left(\frac{1}{100}\right)^2 + \frac{14}{100}\left(\frac{1}{100}\right)^3 + \ldots$$

$r = \frac{1}{100}$ and $|r| < 1$
conv.

$$\text{sum} = \frac{a}{1-r} = \frac{\frac{14}{100}}{1-\frac{1}{100}} = \frac{\frac{14}{100}}{\frac{99}{100}} = \frac{14}{99}$$

Example: Find the values of x so that $\sum_{n=1}^{\infty}(4x-5)^n$ will converge. Find the sum for those values of x.

$$\sum(4x-5)^n = (4x-5)^1 + (4x-5)^2 + (4x-5)^3 + (4x-5)^4 + \cdots$$

$r = 4x-5$

$a = 4x-5$

$|r| < 1$

$|4x-5| < 1$

$-1 < 4x-5 < 1$

$+5 \quad +5 \quad +5$

$4 < 4x < 6$

$1 < x < 6/4 = 3/2$

Interval $\to$ $(1, 3/2)$

$$\text{Sum} = \frac{a}{1-r} = \frac{4x-5}{1-(4x-5)} = \frac{4x-5}{1-4x+5}$$

$$\text{Sum} = \frac{4x-5}{6-4x}$$