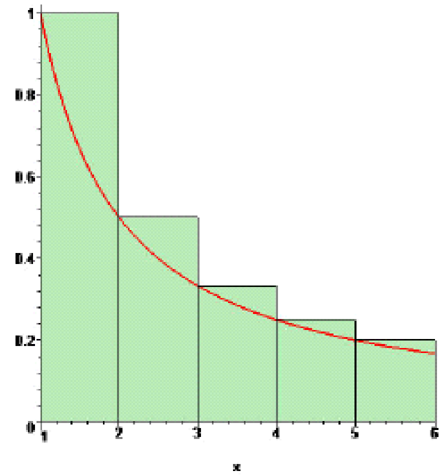
Section 11.3: The Integral Test and Estimates of Sums

Note: In this section all series have positive terms.



This graph was used to discuss the Harmonic series in section 11.2. We saw that

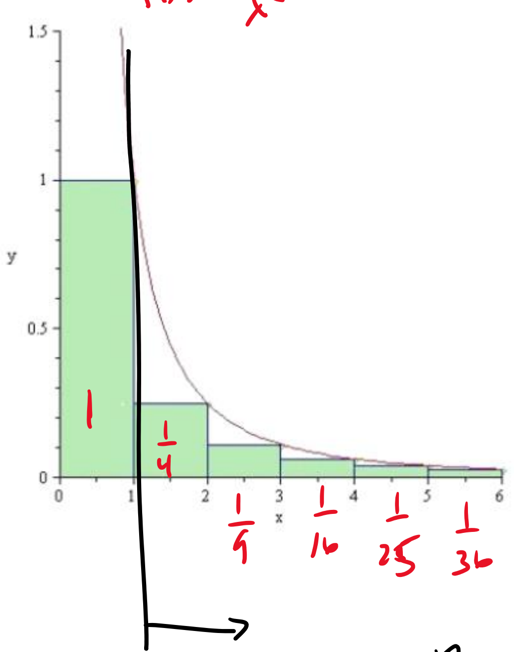
$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\,dx > 0$$

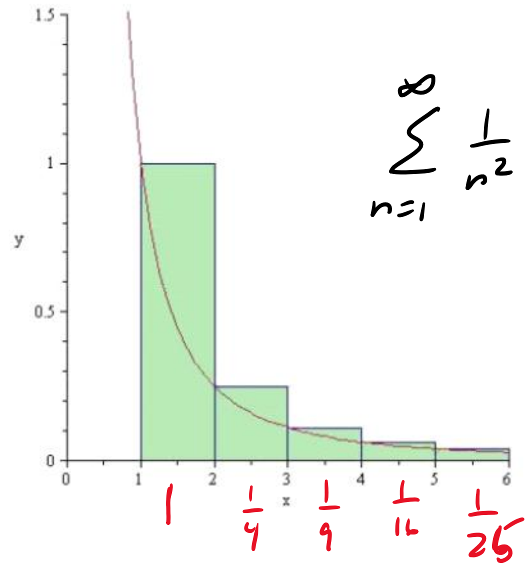
and concluded that the Harmonic series diverged.

div.

Here are the graphs for the function $f(x) = \frac{1}{x^2}$. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

$f(x) = \frac{1}{x^2}$





$$\sum_{n=1}^{\infty} \frac{1}{n^2} > \int_1^{\infty} \frac{1}{x^2}\,dx$$

$$\underbrace{\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx}$$

p integrals.
$p = 2$
Converge.

By comparison we get $\sum_{n=2}^{\infty} \frac{1}{n^2}$ will converge.

Thus $\sum_{n=1}^{\infty} \frac{1}{n^2}$ will conv.

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{matrix} \text{conv } p > 1 \\ \text{div } p \le 1 \end{matrix}$$

p - Integral

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p - series

$$\sum_{n=1}^\infty \frac{1}{n^p} \quad \begin{matrix} \text{conv. } p > 1 \\ \text{div } p \le 1 \end{matrix}$$

The Integral Test Suppose f is **continuous, positive, decreasing function on** $[1,\infty)$, or on $[A,\infty)$, and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(a) If $\int_1^{\infty} f(x)\,dx$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\int_1^{\infty} f(x)\,dx$ is divergent then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example: Assume that $\sum_{n=1}^{\infty} a_n$ is a series where $f(n) = a_n$ and $\int_1^{\infty} f(x)\,dx$ converges to the number L. Does this mean that the series $\sum_{n=1}^{\infty} a_n$ converge to L?

$$\int_1^{\infty} f(x)\,dx = L$$

no.

Test for div.

Example: Do these series converge or diverge?

A) $\sum_{n=1}^{\infty} \frac{n^2}{1+n^2}$

$a_n = \frac{n^2}{1+n^2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{1+n^2} = 1$

By test for div. The series will div.

B) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ $\quad n = 1, 2, 3, \dots$

Test for Div

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

may or may not conv.

Let $f(x) = \dfrac{1}{1+x^2}$ for $x \geq 1$

continuous. ✓

positive ✓

dec. ✓

$$f' = \frac{(1+x^2)\,0 - 1\,(2x)}{(1+x^2)^2}$$

$$f' = \frac{-2x}{(1+x^2)^2} < 0 \text{ for } x \geq 1$$

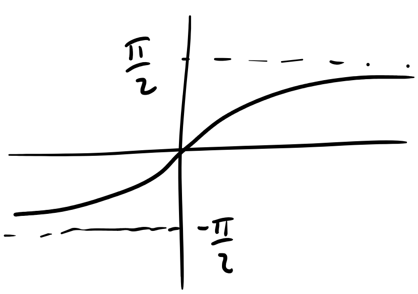
now determine if this integral converges

$$\int_1^{\infty} \frac{1}{1+x^2}\,dx$$

method 1

$$\int_1^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan x \Big|_1^{t}$$

$$= \lim_{t\to\infty} \left(\arctan(t) - \arctan 1\right)$$



$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus the Integral converges.

By Integral test the series will conv.

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method 2)

$$\int_1^{\infty} \frac{1}{1+x^2}\,dx < \underbrace{\int_1^{\infty} \frac{1}{x^2}\,dx}$$

p-integral $p = 2$ convergent

$1+x^2 > x^2$

$$\frac{1}{1+x^2} < \frac{1}{x^2}$$

Thus $\int_1^\infty \frac{1}{1+x^2}\,dx$ will conv.

By the Integral test the series will also conv.

C) $\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n}$

Test for div.

$$\lim_{n\to\infty} \frac{(\ln(n))^2}{n} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{2\ln(n)\cdot\frac{1}{n}}{1}$$

$$= \lim_{n\to\infty} \frac{2\ln(n)}{n} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{2\cdot\frac{1}{n}}{1} = 0$$

May or may not conv.

$f(x) = \frac{(\ln(x))^2}{x}$ $\quad x \geq 1$

$$f'(x) = \frac{x\cdot 2\ln(x)\cdot\frac{1}{x} - (\ln(x))^2\cdot 1}{x^2}$$

$$= \frac{2\ln(x) - (\ln(x))^2}{x^2}$$

$$f'(x) = \frac{\ln(x)\cdot[2-\ln(x)]}{x^2} = 0$$

cont. ✓

dec. $\longrightarrow$ dec for $x > e^2$

positive ✓

$f'(x) = 0$

$\ln(x) = 0$ $\qquad$ $2 - \ln(x) = 0$

$x = 1$ $\qquad$ $2 = \ln(x)$

$e^2 = x$

f': $\times$ $\times$ | $+$ | $-$

1 $\quad e^1$ $\quad e^2$ $\quad e^3$

$$\int_1^\infty \frac{\ln(x)^2}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{(\ln(x))^2}{x}\,dx = \lim_{t\to\infty}\int_0^{\ln(t)} u^2\,du$$

$u = \ln(x)$ $\qquad$ $x = 1 \to u = \ln(1)$

$du = \frac{1}{x}dx$ $\qquad$ $x = t \to u = \ln(t)$

$$= \lim_{t\to\infty} \frac{u^3}{3}\Big|_0^{\ln(t)}$$

$$= \lim_{t\to\infty} \frac{(\ln(t))^3}{3} - 0$$

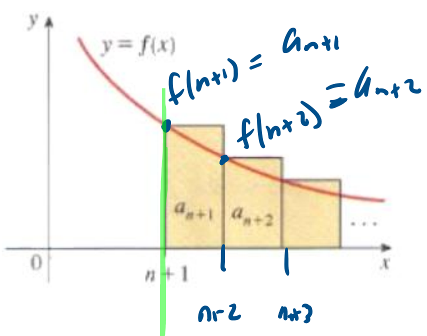
This integral div. $\qquad = \infty$

by The Integral test the series will div.

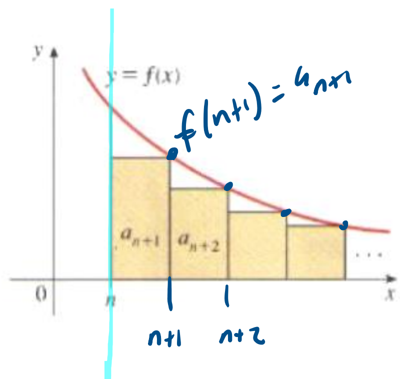
$\lim_{n\to\infty} S_n = S$

Remainder Estimate For The Integral Test Suppose that $\sum a_n$ converges by the Integral Test to the number s. If s_n is a partial sum that approximates this series, define R_n to be the remainder, i.e. $s_n + R_n = s$, then we get the following bounds on the remainder and the sum:

$$\sum_{i=1}^{\infty} a_i = \underbrace{a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n}_{S_n} + \underbrace{a_{n+1} + a_{n+2} + \ldots}_{R_n} = s$$



Left sum



Right sum

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

add S_n to each part

lower bound for the error

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{n}^{\infty} f(x)\,dx$$

notice

$S_n + R_n = \text{Sum}$

upper bound for the error.

Example: Given that this series converges by the Integral test. $\sum_{i=1}^{\infty} \frac{30}{i^4}$.

A) Find the bounds on R_{10}
B) What bounds are on the sum for this series?
C) If you wanted the error to be less than 0.005, what is the smallest value of n should you use?

$$\int_{11}^{\infty} \frac{30}{x^4}\,dx \le R_{10} \le \int_{10}^{\infty} \frac{30}{x^4}\,dx$$

$$\int_{10}^{\infty} \frac{30}{x^4}\,dx = \lim_{t\to\infty} \int_{10}^{t} 30x^{-4}\,dx = \lim_{t\to\infty} \left.\frac{30x^{-3}}{-3}\right|_{10}^{t}$$

$$= \lim_{t\to\infty} \left.\frac{-10}{x^3}\right|_{10}^{t} = \lim_{t\to\infty} \frac{-10}{t^3} - \frac{-10}{(10)^3} = \frac{10}{10^3} = \frac{1}{100}$$

$$\int_{11}^{\infty} \frac{30}{x^4}\,dx = \frac{10}{11^3} \quad \text{by the work above}$$

$$\frac{10}{11^3} \le R_{10} \le \frac{10}{10^3} = \frac{1}{100}$$

to get the bounds on the series compute S_{10}

$S_{10} = 32.4610975$

$$S_{10} + \frac{10}{11^3} \le \text{sum} \le S_{10} + \frac{1}{100}$$

Example: Given that this series converges by the Integral test. $\sum_{i=1}^{\infty} \frac{30}{i^4}$.

A) Find the bounds on R_{10}
B) What bounds are on the sum for this series?
C) If you wanted the error to be less than 0.005, what is the smallest value of n should you use?

find n

$$R_n \le \int_n^{\infty} \frac{30}{x^4}\,dx < .005$$

$$\int_n^{\infty} \frac{30}{x^4}\,dx = \lim_{t\to\infty} \int_n^{t} \frac{30}{x^4}\,dx = \cdots = \frac{10}{n^3} < .005 = \frac{5}{1000} = \frac{1}{200}$$

$$\frac{10}{n^3} < \frac{1}{200}$$

$$10(200) < n^3$$

$$2000 < n^3$$

$$2 \cdot 10^3 < n^3$$

$$12.599 < n \longrightarrow n = 13$$

13 terms

Example: The following series converges by integral test. Determine the smallest number of terms that should be used if the maximum error of the partial sum should be less than 0.05?

$$\sum_{i=1}^{\infty} \frac{1}{i^3}.$$

$$R_n \leq \underbrace{\int_n^{\infty} \frac{1}{x^3}\,dx}_{} < .05 = \frac{5}{100} = \frac{1}{20}$$

$$\lim_{t\to\infty} \int_n^t x^{-3}\,dx = \lim_{t\to\infty} \left.\frac{x^{-2}}{-2}\right|_n^t = \lim_{t\to\infty} \left.-\frac{1}{2x^2}\right|_n^t = \lim_{t\to\infty}\left(\frac{-1}{2t^2} - \frac{-1}{2n^2}\right)$$

$$= \frac{1}{2n^2}$$

$$\frac{1}{2n^2} < \frac{1}{20}$$

$$\frac{20}{2} < n^2$$

$$10 < n^2 \longrightarrow n = 4$$