Section 5.5: The Substitution Rule

Knowing $f(x) = (x^4+3)^4$ and $f'(x) = 4(x^4+3)^3 * 4x^3 = 16x^3(x^4+3)^3$

Compute $\int 16x^3(x^4+3)^3 dx =$ $(x^4+3)^4 + C$

$\int 2x \, dy = 2xy + C$

Example: Compute.

$\frac{1}{9}\int 9 \, 2x(x^2+5)^8 \, dx = \frac{1}{9}(x^2+5)^9 + C$

$\cancel{\frac{1}{9}} \cancel{9} (x^2+5)^8 \cdot 2x$

$du = g'(x)\,dx$

The substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$\int f(g(x))g'(x)\,dx = \int f(u)\,du$

Example: Compute the following. $k \neq 0$.

A) $\int \cos(kx)\,dx = \int \frac{1}{k}\cos(u)\,du = \frac{1}{k}\sin(u) + C$

$= \frac{1}{k}\sin(kx) + C$

$u = kx$

$du = k\,dx$

$\frac{1}{k}du = dx$

$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

$\int \sin(kx)\,dx = \frac{-1}{k}\cos(kx) + C$

B) $\int \frac{12x^3+9}{(x^4+3x)^5}\,dx \quad = \int \frac{3(4x^3+3)}{(x^4+3x)^5}\,dx$

$u = x^4 + 3x$

$du = (4x^3+3)\,dx$

$= \int \frac{3}{u^5}\,du$

$\frac{1}{4x^3+3}\,du = dx$

$\int \frac{12x^3+9}{u^5}\,\frac{1}{4x^3+3}\,du = \int \frac{3(4x^3+3)}{u^5}\,\frac{1}{4x^3+3}\,du$

$= \int \frac{3}{u^5}\,du = \int 3u^{-5}\,du = \frac{3u^{-4}}{-4} + C$

$= -\frac{3}{4u^4} + C = \frac{-3}{4(x^4+3x)^4} + C$

u sub

u = junk

$(\text{junk})^{\text{power}}$

Trig (junk)

e^{junk}

$\frac{\text{something}}{\text{junk}}$

C) $\int x(x-8)^8\,dx = \int x\,u^8\,du = \int (u+8)\,u^8\,du$

$\boxed{u = x-8}$ $\rightarrow$ $\underbrace{u+8}=x$

$du = dx$

$$= \int u^9 + 8u^8\,du$$

$$= \frac{u^{10}}{10} + \frac{8u^9}{9} + C$$

$$= \frac{(x-8)^{10}}{10} + \frac{8}{9}(x-8)^9 + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x)$$

D) $\displaystyle \underbrace{\int \frac{1+4x}{1+x^2}\,dx}$ $= \cancel{\displaystyle\int \frac{1+4x}{u} \cdot \frac{1}{2x}\,du}$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2x}\,du = dx$

$$\int \frac{4x}{1+x^2} + \frac{1}{1+x^2}\,dx$$

$$\int \frac{4x}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$$

$$= \int \frac{4x}{u} \cdot \frac{1}{2x}\,du + \arctan(x) + C$$

$$= \int \frac{2}{u}\,du + \arctan(x) + C$$

$$= 2\ln|u| + \arctan(x) + C$$

$$= 2\ln|1+x^2| + \arctan(x) + C$$

The substitution Rule for Definite Integrals If $g'(x)$ is differentiable on $[a, b]$ and f is continuous on the range of g, then continuous on I, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$u = g(x)$

$du = g'(x)\,dx$

$\frac{x=a}{u=g(a)}$ $\frac{x=b}{u=g(b)}$

Example: Compute

$$\int_0^2 x\cos(4x^2-1)dx = \int_{-1}^{15} \frac{1}{8}\cos(u)\,du = \frac{1}{8}\sin(u)\Big|_{-1}^{15}$$

$u = 4x^2 - 1$

$du = 8x\,dx$

$\frac{1}{8}du = x\,dx$

$\underline{x=0}$ $u = 4(0)^2 - 1 = -1$

$\underline{x=2}$ $u = 4(2)^2 - 1 = 16 - 1 = 15$

$\frac{1}{8}\sin(15) - \frac{1}{8}\sin(-1)$

$$\int_0^2 x\cos(4x^2-1)\,dx = \int_{x=0}^{x=2} \frac{1}{8}\cos(u)\,du = \frac{1}{8}\sin(u)\Big|_{x=0}^{x=2}$$

$$= \frac{1}{8}\sin(4x^2-1)\Big|_0^2$$

$$= \frac{1}{8}\sin(15) - \frac{1}{8}\sin(-1)$$

Example: Compute

$$\int_0^3 2x^3(1-x^2)^5\,dx = \int_{x=0}^{x=3} 2x^3 u^5 \frac{-1}{2x}\,du = \int_{x=0}^{x=3} -x^2 u^5\,du$$

$$\boxed{u = 1 - x^2} \qquad \boxed{x^2 = 1-u}$$

$du = -2x\,dx$

$-\frac{1}{2x}\,du = dx$

$x=0$
$u=1$

$x=3$
$u=-8$

$$= \int_1^{-8} -(1-u)u^5\,du$$

$$= \int_1^{-8} -u^5 + u^6\,du$$

$$= \left(-\frac{u^6}{6} + \frac{u^7}{7}\right)\Bigg|_1^{-8}$$

$$= -\frac{(-8)^6}{6} + \frac{(-8)^7}{7} - \left(-\frac{1}{6} + \frac{1}{7}\right)$$

$$= -\frac{8^6}{6} - \frac{8^7}{7} + \frac{1}{6} - \frac{1}{7}$$