Section 7.1: Integration by Parts

Product rule: $\frac{d}{dx}(f * g) = f' * g + f * g'$

$$\frac{d}{dx}(fg) = f'g + fg'$$

$$\int \frac{d}{dx}(fg)\,dx = \int f'g\,dx + \int fg'\,dx$$

$$fg = \int f'g\,dx + \int fg'\,dx$$

$$\int fg'\,dx = fg - \int f'g\,dx$$

$u = f$

$dv = g'$

$$\int u\,dv = uv - \int v\,du$$

$\int u\,dv = uv - \int v\,du$

Example: Compute the following integrals.

A) $\int xe^{2x}\,dx$ $= \underbrace{\frac{x^2}{2} e^{2x} - \int \frac{x^2}{2} 2e^{2x}\,dx}$

$u = e^{2x}$ $\quad du = 2e^{2x}$

$dv = x$ $\quad v = \frac{x^2}{2}$

not a nicer Int.

Bad choice

$$\int xe^{2x}\,dx = x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x} \cdot 1\,dx$$

$u = x$ $\quad du = 1\,dx$

$dv = e^{2x}\,dx$ $\quad v = \frac{1}{2}e^{2x}$

$$= \frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$= \underbrace{\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C}$$

$$\int_0^3 xe^{2x}\,dx = \left(\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right)\Big|_0^3$$

$$= \frac{3}{2}e^6 - \frac{1}{4}e^6 - \left(0 - \frac{1}{4}e^0\right) = \frac{3}{2}e^6 - \frac{1}{4}e^6 + \frac{1}{4}$$

B) $\int x^2 e^{2x}\,dx$

$\int u\,dv = uv - \int v\,du$

$u = x^2 \qquad dv = e^{2x}$

$du = 2x \qquad v = \frac{1}{2}e^{2x}$

$$\int x^2 e^{2x}\,dx = x^2 \cdot \frac{1}{2}e^{2x} - \int 2x \cdot \frac{1}{2}e^{2x}\,dx$$

$$= \frac{x^2}{2}e^{2x} - \int x e^{2x}\,dx$$

$u = x \qquad dv = e^{2x}$

$du = 1 \qquad v = \frac{1}{2}e^{2x}$

$$= \frac{x^2}{2}e^{2x} - \left[x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx\right]$$

$$= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \int \frac{1}{2}e^{2x}\,dx$$

$$= \underbrace{\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C}$$

Page 4: tabular method

$\int u\,dv = uv - \int v\,du$

Example: Compute the following integrals. Tabular Method Used.

A) $\int xe^{2x}\,dx$ $= x \cdot \frac{1}{2}e^{2x} - \underbrace{\int \frac{1}{2}e^{2x}\,dx} = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$

D		I	
$u = x$	$+$	e^{2x}	$= dv$
$du = 1$	$-\int$	$\frac{1}{2}e^{2x}$	$= v$

D		I
x	$+$	e^{2x}
1	$-$	$\frac{1}{2}e^{2x}$
0	$+\int$	$\frac{1}{4}e^{2x}$

$$\int xe^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + \int 0\,dx$$

$$= \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$$

B) $\int x^2 e^{2x}\, dx \quad = \quad \frac{x^2}{2} e^{2x} - \frac{2x}{4} e^{2x} + \frac{2}{8} e^{2x} + C$

D		I
x^2	$+$	e^{2x}
$2x$	$-$	$\frac{1}{2} e^{2x}$
2	$+$	$\frac{1}{4} e^{2x}$
0	$-\int$	$\frac{1}{8} e^{2x}$

C) $\int (x^3 + 6x)\sin(2x)\, dx$

$\int x \sin(x^2)\, dx$

u-sub.

D		I
x^3+6x	$+$	$\sin(2x)$
$3x^2+6$	$-$	$-\frac{1}{2}\cos(2x)$
$6x$	$+$	$-\frac{1}{4}\sin(2x)$
6	$-$	$\frac{1}{8}\cos(2x)$
0	$+\int$	$\frac{1}{16}\sin(2x)$

$$\int (x^3+6x)\sin(2x)\,dx = (x^3+6x)\left(\frac{-1}{2}\right)\cos(2x) + \frac{3x^2+6}{4}\sin(2x) + \frac{6x}{8}\cos(2x) - \frac{6}{16}\sin(2x) + C$$

D) $\int 1 \ln(x)\, dx = x\ln(x) - \int \frac{1}{x} x\, dx = x\ln(x) - \int 1\, dx$

$= x\ln(x) - x + C$

D		I
$\ln(x)$	$+$	1
$\frac{1}{x}$	$-\int$	x

$\ln x$	$+$	1
$\frac{1}{x}$	$-$	x
$\frac{-1}{x^2}$	$+\int$	$\frac{x^2}{2}$

$x\ln(x) - \frac{1}{x}\cdot\frac{x^2}{2} + \int \frac{-1}{x^2}\cdot\frac{x^2}{2}\, dx$

$x\ln(x) - \frac{1}{2}x - \int \frac{1}{2}\, dx$

$x\ln(x) - \frac{1}{2}x - \frac{1}{2}x + C = x\ln(x) - x + C$

$$2\ln(x)\cdot\frac{1}{x}\cdot\frac{x^3}{3}$$

E) $\int x^2(\ln(x))^2\,dx \quad = \quad \frac{x^3}{3}(\ln(x))^2 - \int \frac{2}{3}\ln(x)\cdot x^2\,dx$

D		I
$(\ln(x))^2$	$+$	x^2
$2\ln(x)\cdot\frac{1}{x}$	$-\int$	$\frac{x^3}{3}$

D		I
$\frac{2}{3}\ln(x)$	$+$	x^2
$\frac{2}{3x}$	$-\int$	$\frac{x^3}{3}$

$$\frac{x^3}{3}(\ln(x))^2 - \left[\frac{2}{3}\cdot\frac{x^3}{3}\ln(x) - \int\frac{2x^2}{9}\,dx\right]$$

$$\frac{x^3}{3}(\ln(x))^2 - \frac{2}{9}x^3\ln(x) + \frac{2}{9}\cdot\frac{x^3}{3} + C$$

F) $\int x\tan^2(x)\,dx$

D	I
x	$\tan^2(x) = \sec^2 x - 1$
1	$\tan(x) - x = \frac{\sin(x)}{\cos x} - x$
$-\int$	

$+$

$\cos^2\theta + \sin^2\theta = 1$

$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$

$1 + \tan^2\theta = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$x(\tan(x) - x) - \int \frac{\sin(x)}{\cos x} - x \; dx$

$x(\tan(x) - x) - \int \frac{\sin(x)}{\cos(x)}\,dx + \int x\,dx$

$\int \frac{\sin(x)}{\cos(x)}\,dx = \int -\frac{1}{u}\,du$

$u = \cos(x)$

$du = -\sin(x)dx$

$= -1\ln|u|$

$= -\ln|\cos(x)|$

$x(\tan(x) - x) + \ln|\cos x| + \frac{x^2}{2} + C$

!!

G) $\int 1 \arctan(x)\, dx \quad = \quad x \arctan(x) - \underbrace{\int \frac{x}{1+x^2}\, dx}_{}$

u-sub.

D	I
$\arctan(x)$	1
$\frac{1}{1+x^2}$	x

$+$

$-\int$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\int \frac{x}{1+x^2}\,dx$

$= \int \frac{1}{2}\frac{1}{u}\,du$

$= \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln|1+x^2|$

$x\arctan(x) - \frac{1}{2}\ln|1+x^2| + C$

H) $\int \underline{\sin(x)\cos(3x)}\,dx = -\cos(x)\cos(3x) - 3\sin(x)\sin(3x) + \int 9\sin(x)\cos(3x)\,dx$

D		I
$\cos(3x)$	$+$	$\sin x$
$-3\sin(3x)$	$-$	$-\cos x$
$-9\cos(3x)$	$+\int$	$-\sin(x)$

Let $J = \int \sin(x)\cos(3x)\,dx$

$$J = \underbrace{-\cos(x)\cos(3x) - 3\sin(x)\sin(3x)}_{} + 9J$$

$$-8J = [\qquad\qquad]$$

$$J = -\frac{1}{8}\left[-\cos(x)\cos(3x) - 3\sin(x)\sin(3x)\right] + C$$

I) $\int \sin(x)e^{3x}\,dx = J$

D		I
e^{3x}	$+$	$\sin(x)$
$3e^{3x}$	$-$	$-\cos x$
$9e^{3x}$	$+\int$	$-\sin(x)$

$$J = -\cos(x)e^{3x} + 3\sin(x)e^{3x} - 9\int \sin(x)\, e^{3x}\,dx$$

$$J = \underbrace{-\cos(x)e^{3x} + 3\sin(x)e^{3x}} - 9J$$

$$10J = [\quad]$$

$$J = \frac{1}{10}\left[-\cos(x)e^{3x} + 3\sin(x)e^{3x}\right] + C$$

J) $\int x^3 \sin(x^2)\, dx$

since this is a $\sin(x^2)$ lets do a u-sub.

$u = x^2$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$\int x^3 \sin(x^2)\, dx = \int x^2 \sin(u) \cdot \frac{1}{2}\, du = \underbrace{\int \frac{u}{2} \sin(u)\, du}$$

now we can do Int. by parts.

D	I
$\frac{u}{2}$	$\sin(u)$
$\frac{1}{2}$	$-\cos(u)$
0	$-\sin(u)$

$-$ $+$ $-\int$

$$= \frac{u}{2}\cos(u) - \frac{1}{2}\sin(u) + C$$

final Answer.

$$= \frac{x^2}{2}\cos(x^2) - \frac{1}{2}\sin(x^2) + C$$