Section 7.2: Trig Integrals

Extremly Useful Trig Identities

$$\boxed{\begin{aligned}\sin^2(x) &= \tfrac{1}{2}(1-\cos(2x))\\ \cos^2(x) &= \tfrac{1}{2}(1+\cos(2x))\end{aligned}}$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\sin^2(x)+\cos^2(x) = 1$$

$$\tan^2(x)+1 = \sec^2(x)$$

Sometimes Useful Trig Identities

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$$

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2(x) + 1 = \sec^2(x)$$

$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

Compute these integrals

Example: $\int \sin(5x)\sin(4x)\ dx$

$$= \int \frac{1}{2}\left(\cos(5x-4x) - \cos(5x+4x)\right)\ dx$$

$$= \int \frac{1}{2}\left(\cos(x) - \cos(9x)\right)\ dx$$

$$= \frac{1}{2}\left[\sin(x) - \frac{1}{9}\sin(9x)\right] + C$$

Example: $\int \sin(7x)\cos^4(7x)\ dx$ $= \int \sin(7x)\left[\cos(7x)\right]^4 dx = \int -\frac{1}{7} u^4\ du$

$u = \cos(7x)$

$du = -\sin(7x)\cdot 7\ dx$

$\underbrace{-\frac{1}{7}\,du} = \underline{\sin(7x)\,dx}$

$= -\frac{1}{7}\cdot\frac{1}{5}u^5 + C$

$= \frac{-1}{35}\cos^5(7x) + C$

Page 4: u-subs

$$\int \sin^m(x)\cos(x)dx$$

$u = \sin(x)$

$$\int \sec^m(x)\,\underline{\sec(x)\tan(x)dx}$$

$u = \sec x$

$$\int \cos^m(x)\sin(x)dx$$

$u = \cos(x)$

$$\int \cot^m(x)\csc^2(x)dx$$

$u = \cot x$

$$\int \tan^m(x)\sec^2(x)dx$$

$u = \tan(x)$

$$\int \csc^m(x)\csc(x)\cot(x)dx$$

$u = \csc x$

Example: $\int \sin^2(3x)\ dx$

$\sin^2\theta + \cos^2\theta = 1$

$\cancel{\int 1 - \cos^2(3x)\ dx}$

$\sin^2\theta = \frac{1}{2}\left(1 - \cos(2\theta)\right)$

$= \int \frac{1}{2}\left(1 - \cos(6x)\right) dx$

$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right] + C$

$= \frac{x}{2} - \frac{1}{12}\sin(6x) + C$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

Example: $\int \cos^4(x)\,dx = \int \cos^2 x \cos^2 x \, dx$

$$= \int \frac{1}{2}\underbrace{(1+\cos 2x)} \cdot \frac{1}{2}\underbrace{(1+\cos 2x)}\,dx$$

$$= \frac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)\,dx$$

$$= \frac{1}{4}\int 1 + 2\cos(2x) + \frac{1}{2}(1+\cos(4x))\,dx$$

$$= \frac{1}{4}\int 1 + 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x)\,dx$$

$$= \frac{1}{4}\int \frac{3}{2} + 2\cos(2x) + \frac{1}{2}\cos(4x)\,dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \frac{2\sin(2x)}{2} + \frac{1}{2}\cdot\frac{1}{4}\sin(4x)\right] + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

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u-sub

$$\int \cos^4(x)\,dx = \int \frac{-1}{\sin(x)}u^4\,du$$

not a valid method.

$u = \cos(x)$

$du = -\sin(x)\,dx$

$\frac{-du}{\sin(x)} = dx$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$\sin^2\theta = 1 - \cos^2\theta$$

Example: $\int \sin^5(2x)\,dx = \int \sin^4(2x)\cdot\sin(2x)\,dx$

$$= \int \sin^2(2x)\cdot\sin^2(2x)\,\sin(2x)\,dx$$

$$= \int \left(1-\cos^2(2x)\right)\left(1-\cos^2(2x)\right)\sin(2x)\,dx$$

$$= \int \left(1-\cos^2(2x)\right)^2 \sin(2x)\,dx = \int -\frac{1}{2}\left(1-u^2\right)^2\,du$$

$u = \cos(2x)$

$du = -2\sin(2x)\,dx$

$-\frac{1}{2}\,du = \sin(2x)\,dx$

$$= -\frac{1}{2}\int 1 - 2u^2 + u^4\,du$$

$$= -\frac{1}{2}\left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right] + C$$

$$= -\frac{1}{2}\left[\cos(2x) - \frac{2}{3}\cos^3(2x) + \frac{1}{5}\cos^5(2x)\right] + C$$

$\cos^3(2x)$ ✓ $\quad \cos(2x)^3$ ✗ $\quad \left(\cos(2x)\right)^3$ ✓

Example: $\int \sin^4(3x)\cos^3(3x)\,dx = \int \sin^4(3x)\,\cos^2(3x)\,\cos(3x)\,dx$

$$= \int \sin^4(3x)\left(1-\sin^2(3x)\right)\cos(3x)\,dx$$

$u = \sin(3x)$

$du = 3\cos(3x)\,dx$

$\frac{1}{3}du = \cos(3x)\,dx$

$$= \frac{1}{3}\int u^4(1-u^2)\,du$$

$$= \frac{1}{3}\int u^4 - u^6\,du$$

$$= \frac{1}{3}\left[\frac{u^5}{5} - \frac{u^7}{7}\right] + C$$

$$= \frac{1}{3}\left[\frac{1}{5}\sin^5(3x) - \frac{1}{7}\sin^7(3x)\right] + C$$

Example: $\int \sec^4(x)\,dx$

$$= \int \underbrace{\sec^2(x)}\ \underline{\sec^2(x)}\ dx$$

convert to $\tan^2 x$?

$$\boxed{\sec^2\theta = \tan^2\theta + 1}$$

$$= \int (\tan^2 x + 1)\ \sec^2 x\ dx$$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$$= \int u^2 + 1\ du = \frac{1}{3}u^3 + u + C$$

$$= \frac{1}{3}\tan^3(x) + \tan x + C$$

Bad method

$$\int \frac{1}{\cos^4 x}\,dx$$

$$\int \frac{1}{\cos^2 x\ \cos^2 x}\,dx$$

$$\cancel{\int \frac{1}{\frac{1}{2}(1+\cos(2x))\cdot\frac{1}{2}(1+\cos(2x))}\,dx}$$

$\tan^2\theta + 1 = \sec^2\theta$

Example: $\int \tan^4(x)\sec^4(x)\,dx$

$u = \tan(x)$

$$\int \underbrace{\tan^4 x \sec^2 x}\ \underline{\sec^2 x}\, dx$$

$$\int \tan^4 x\,(\tan^2 x + 1)\sec^2 x\, dx$$

$u = \sec x$

$$\int \underbrace{\tan^3 x \sec^3 x}\ \underline{\sec x \tan x}\, dx$$

method not good

Since $\tan^3 x$ is odd power

$u = \tan(x)$

$du = \sec^2 x\, dx$

$$= \int u^4(u^2+1)\,du = \int u^6 + u^4\,du = \frac{u^7}{7} + \frac{u^5}{5} + C$$

$$= \frac{1}{7}\tan^7(x) + \frac{1}{5}\tan^5(x) + C$$

Example: $\int \tan^5(x)\sec^3(x)\ dx$

$u = \sec(x)$

$du = \sec(x)\tan(x)\ dx$

$\cancel{}\ \tan^5(x)\ \sec(x)\ \sec^2(x)$

$$\int \tan^4 x\ \sec^2 x\ \sec x \tan x\ dx$$

$$= \int \tan^2 x\ \tan^2 x\ \sec^2 x\ \sec x \tan x\ dx$$

$$= \int (\sec^2 x - 1)^2\ \sec^2 x\ \sec x\ \tan x\ dx = \int (u^2-1)^2 u^2\ du$$

$$= \int (u^4 - 2u^2 + 1)\ u^2\ du = \int u^6 - 2u^4 + u^2\ du$$

$$= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C$$

$$= \frac{1}{7}\sec^7(x) - \frac{2}{5}\sec^5(x) + \frac{1}{3}\sec^3 x + C$$

Example: $\int \sec(x)\,dx$ $= \displaystyle\int \sec(x) \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec(x)}\,dx = \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$u = \tan x + \sec x$

$du = (\sec^2 x + \sec x \tan x)\,dx$

$$\boxed{\int \sec x\,dx = \ln|\tan x + \sec x| + C}$$

Example: $\int \sec^3(x)\,dx = \int \sec(x)\,\sec^2(x)\,dx$

D		I
$\sec x$	$+$	$\sec^2 x$
$\sec x \tan x$	$-\int$	$\tan x$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x\,(\sec^2 x - 1)\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x - \sec x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$

Example: $\int \cot^2(x)\csc^2(x)\,dx$ $= \int -u^2\,du = -\frac{1}{3}u^3 + C$

$u = \cot x$

$= -\frac{1}{3}\cot^3 x + C$

$du = -\csc^2 x\,dx$

$-du = \csc^2 x\,dx$

$$\int \tan^3 x\, dx = \int \tan x \tan^2 x\, dx = \int \tan x\,(\sec^2 x - 1)\, dx$$

$$= \int \underbrace{\tan x \sec^2 x}\ - \tan x\, dx$$

$u = \tan x$

$du = \sec^2 x\, dx$

$$= \int \tan x \sec^2 x\, dx - \int \tan x\, dx$$

$$= \int u\, du - \int \frac{\sin x}{\cos x}\, dx$$

$u = \cos(x)$

$du = -\sin(x)\cancel{dx}$

$-du = \sin(x)\, dx$

$$= \frac{1}{2}u^2 - \int -\frac{1}{u}\, du$$

$$= \frac{1}{2}\tan^2 x + \int \frac{1}{u}\, du$$

$$+ \ln|u|$$

$$= \underbrace{\frac{1}{2}\tan 2x + \ln|\cos x| + C}$$

$$\frac{1}{2}\tan^2 x - \ln|\sec x| + C$$

$$-\ln|u|$$

$$-\ln|\cos x| = \ln|(\cos x)^{-1}| = \ln\left|\frac{1}{\cos x}\right| = \ln|\sec x|$$

$$\int \tan^3 x \sec^4 x \, dx$$

$u = \tan x$

$$\int \tan^3 x \sec^2 x \sec^2 x \, dx$$

$$\int \tan^3 x (\tan^2 x + 1) \sec^2 x \, dx$$

$$\int u^3 (u^2+1) \, du$$

$$\int u^5 + u^3 \, du$$

$$= \frac{1}{6}u^6 + \frac{1}{4}u^4 + C$$

$$\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C$$

$u = \sec x$

$$\int \tan^2 x \sec^3 x \sec x \tan x \, dx$$

$$\int (\sec^2 x - 1) \sec^3 x \sec x \tan x \, dx$$

$$\int (u^2 - 1) u^3 \, du$$

$$\int u^5 - u^3 \, du$$

$$\frac{1}{6}u^6 - \frac{1}{4}u^4 + C$$

$$\frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C$$