Section 7.3: Trigonometric Substitution

Comparison of two integrals.

$$\int x\sqrt{1-x^2}\,dx = \int \frac{-1}{2}u^{1/2}du = \frac{-1}{2} * \frac{2}{3}u^{3/2} + C = \frac{-1}{3}(1-x^2)^{3/2} + C$$

$u = 1 - x^2$ $\qquad$ $\frac{-1}{2}du = x\,dx$

$$\int \sqrt{1-x^2}\,dx$$

$$\int (1-x^2)^2\,dx$$

Examine: $\sqrt{1-x^2}$

$\sin^2\theta + \cos^2\theta = 1$

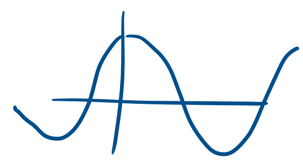
$\cos^2\theta = 1 - \sin^2\theta$

If $x = \sin\theta$

Let $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

$\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$

$= \cos\theta$



Some useful integrals.

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$
$$\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$
$$\int \csc^3 x \, dx = \frac{-1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$$

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \ \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \ \frac{-\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Compute these integrals.

Example: $\int \sqrt{16-x^2}\,dx$

$$\cos^2\theta = 1-\sin^2\theta$$

$$\sec^2\theta = 1+\tan^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1$$

$$16\cos^2\theta = 16-16\sin^2\theta$$

Let $x^2 = 16\sin^2\theta$

$$\boxed{x = 4\sin\theta} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$dx = 4\cos\theta\,d\theta$$

$$\int \sqrt{16-x^2}\,dx = \int \sqrt{16-16\sin^2\theta}\;4\cos\theta\,d\theta = \int \sqrt{16\cos^2\theta}\cdot 4\cos\theta\,d\theta$$

$$= \int 4\cos\theta\cdot 4\cos\theta\,d\theta = \int 16\cos^2\theta\,d\theta$$

$$= 16\int \cos^2\theta\,d\theta = 16\cdot\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 8\int 1+\cos 2\theta\,d\theta$$

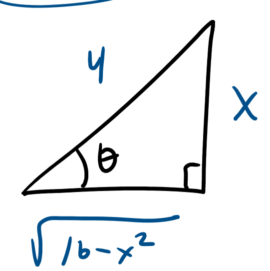
$$= 8\cdot\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$= 8\left[\theta + \frac{1}{2}\cdot 2\sin\theta\cos\theta\right] + C$$

$$= 8\left[\arcsin\left(\frac{x}{4}\right) + \frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4}\right] + C$$

$x = 4\sin\theta$

$\frac{x}{4} = \sin\theta \qquad \theta = \arcsin\left(\frac{x}{4}\right)$



$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos\theta = \frac{\sqrt{16-x^2}}{4}$

not complty correct.

$$8\left[\arctan\left(\frac{x}{4}\right) + \frac{1}{2}\sin\left(2\arctan\frac{x}{4}\right)\right] + C$$

Example: $\int \frac{1}{(x^2-9)^{3/2}}\,dx$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1$$

$$9\tan^2\theta = 9\sec^2\theta - 9$$

need $x^2 = 9\sec^2\theta$

Let $x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$$\int \frac{1}{(x^2-9)^{3/2}}\,dx = \int \frac{1}{(9\tan^2\theta)^{3/2}} \cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{3\sec\theta\tan\theta}{9^{3/2}(\tan^2\theta)^{3/2}}\,d\theta = \int \frac{3\sec\theta\tan\theta}{3^3\tan^3\theta}\,d\theta$$

$$= \frac{1}{3^2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{9}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta$$

$$= \frac{1}{9}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u = \sin\theta \qquad du = \cos\theta\,d\theta$

$$= \frac{1}{9}\int \frac{1}{u^2}\,du = \frac{1}{9}\int u^{-2}\,du = \frac{1}{9}\cdot\frac{u^{-1}}{-1} + C$$

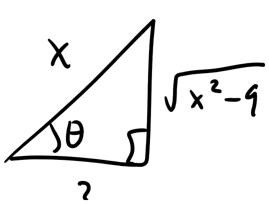
$$= -\frac{1}{9u} + C$$

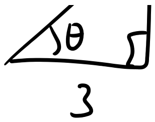
$$= -\frac{1}{9\sin\theta} + C$$

$$= \frac{-1}{9\,\frac{\sqrt{x^2-9}}{x}} + C = -\frac{x}{9\sqrt{x^2-9}} + C$$

$X = 3\sec\theta$

$\frac{x}{3} = \sec\theta \qquad \cos\theta = \frac{3}{x}$





$$\sin\theta = \frac{\sqrt{x^2-9}}{x}$$

$$= \frac{-1}{9\,\frac{\sqrt{x^2-9}}{x}} + C = -\frac{x}{9\sqrt{x^2-9}} + C$$

Example: $\int \frac{1}{x^2\sqrt{16-9x^2}}\,dx$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$16\cos^2\theta = 16 - 16\sin^2\theta$

need $\underbrace{9x^2 = 16\sin^2\theta}$

Let $3x = 4\sin\theta$

$x = \frac{4}{3}\sin\theta$

$dx = \frac{4}{3}\cos\theta\, d\theta$

$$\int \frac{1}{x^2\sqrt{16-9x^2}}\,dx$$

$$= \int \frac{1}{\frac{16}{9}\sin^2\theta\sqrt{16\cos^2\theta}} \cdot \frac{4}{3}\cos\theta\, d\theta$$

$$= \int \frac{1}{\frac{16}{9}\sin^2\theta \cdot 4\cos\theta} \cdot \frac{4}{3}\cos\theta\, d\theta$$

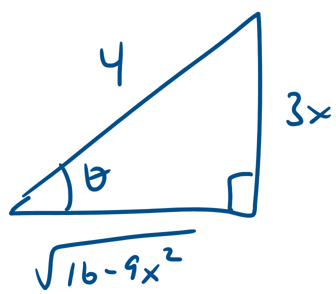
$$= \int \frac{9}{16}\cdot\frac{1}{\sin^2\theta}\cdot\frac{1}{3}\,d\theta = \int \frac{3}{16}\frac{1}{\sin^2\theta}\,d\theta$$

$$= \frac{3}{16}\int \csc^2\theta\, d\theta = -\frac{3}{16}\cot\theta + C$$

$$= \frac{-3}{16}\cdot\frac{\sqrt{16-9x^2}}{3x} + C$$

$3x = 4\sin\theta$

$\frac{3x}{4} = \sin\theta$



$\tan\theta = \frac{3x}{\sqrt{16-9x^2}}$

$\cot\theta = \frac{\sqrt{16-9x^2}}{3x}$

Example: $\int \frac{1}{x^2 + A^2}\, dx$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$A^2\sec^2\theta = A^2 + A^2\tan^2\theta$

need $x^2 = A^2\tan^2\theta$

let $x = A\tan\theta$

$dx = A\sec^2\theta\, d\theta$

$\frac{x}{A} = \tan\theta$

$\theta = \arctan\left(\frac{x}{A}\right)$

$$\int \frac{1}{x^2+A^2}\, dx = \int \frac{1}{A^2\sec^2\theta} \cdot A\sec^2\theta\, d\theta$$

$$= \int \frac{1}{A}\, d\theta = \frac{1}{A}\theta + C$$

$$\boxed{\int \frac{1}{x^2+A^2}\, dx = \frac{1}{A}\arctan\left(\frac{x}{A}\right) + C}$$

Page 7: Complete the square.

Review of completing the square:

$x^2 + 8x =$ $\quad x^2 + 8x + 4^2 - 4^2 = \underbrace{x^2 + 8x + 16}_{} - 16$

1) make the # in front of x^2 a one.

$(x+4)^2 - 16$

2) take $\frac{1}{2}$ of the # in front of the x. $\to \frac{1}{2}(8) = 4$

3) add + subtract the square of this # to the expression.

4) factor.

$4x^2 + 24x + 11 =$ $4[x^2 + 6x] + 11$

$= 4[x^2 + 6x + 3^2 - 3^2] + 11$

$= 4[\underbrace{x^2 + 6x + 9}_{} - 9] + 11$

$= 4[(x+3)^2 - 9] + 11 = 4(x+3)^2 - 36 + 11$

$= 4(x+3)^2 - 25$

Example: $\int \frac{x}{\sqrt{4x-x^2}}\,dx$

Step 1 complete the square

$$4x - x^2 = -x^2+4x = -1\left[x^2-4x\right] = -1\left[x^2-4x+2^2-2^2\right]$$

$$= -1\left[x^2-4x+4-4\right] = -1\left[(x-2)^2-4\right]$$

$$= 4-(x-2)^2$$

Example: $\int \frac{x}{\sqrt{4x-x^2}}\,dx = \int \frac{x}{\sqrt{4-(x-2)^2}}\,dx$

$\cos^2\theta = 1-\sin^2\theta$

$\sec^2\theta = 1+\tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$4\cos^2\theta = 4 - 4\sin^2\theta$

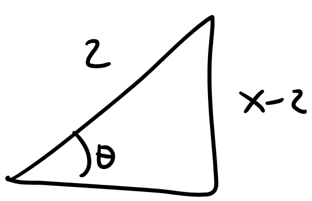
need $4\sin^2\theta = (x-2)^2$

Let $x-2 = 2\sin\theta$

$x = 2+2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$\frac{x-2}{2} = \sin\theta$



$$\int \frac{2+2\sin\theta}{\sqrt{4\cos^2\theta}} \cdot 2\cos\theta\, d\theta$$

$$= \int \frac{2+2\sin\theta}{2\cos\theta} \cdot 2\cos\theta\, d\theta$$

$$= \int 2+2\sin\theta\, d\theta = 2\theta - 2\cos\theta + C$$

$$= 2\arcsin\left(\frac{x-2}{2}\right) - 2\,\frac{\sqrt{4-(x-2)^2}}{2} + C$$

$\sqrt{2^2-(x-2)^2}$ $\sqrt{4-(\ldots)^2}$

$\sqrt{2^2-(x-2)^2} = \sqrt{4-(x-2)^2}$

OR

$2\arcsin\left(\frac{x-2}{2}\right) - \sqrt{4x-x^2} + C$