$$\frac{3}{x+2}+\frac{4}{x+5}=\frac{3(x+5)+4(x+2)}{(x+2)(x+5)}=\frac{7x+23}{x^2+7x+10}$$

$$\int \frac{7x+23}{x^2+7x+10}\,dx=\int \frac{3}{x+2}+\frac{4}{x+5}\,dx=\int \frac{3}{x+2}\,dx+\int\frac{4}{x+5}\,dx=\cdots=3\ln|x+2|+4\ln|x+5|+C$$

$u=x+2$ $\quad$ $u=x+5$

A rational function is a function of the form $\frac{P(x)}{Q(x)}$ where both $P(x)$ and $Q(x)$ are polynomials. The degree of a polynomial is the highest power of the variable.

NOTE: To integrate a rational function, $\frac{P(x)}{Q(x)}$, with the partial fraction method, you MUST HAVE the degree $P(x)$ < degree $Q(x)$. If this is not the case then use long division(or some other method) to find $J(x)$ and $K(x)$ so that

$$\frac{P(x)}{Q(x)}=J(x)+\frac{K(x)}{Q(x)}$$

Method of Integration by Partial Fractions:

0) Do long division(or other algebra manipulation) if degree $P(x)\geq$ degree $Q(x)$.
1) Factor the denominator completely
2) Decompose the fraction
3) Solve for the constants in the decomposition
4) Integrate the new fractions

Example: Compute these Integrals.

A) $\displaystyle\int \frac{x^3+2x^2-5}{x+1}\,dx$

$$
\begin{array}{r|l}
 & \;x^2 + x \quad -1 \\
\hline
x+1 & x^3 + 2x^2 + 0x - 5 \\
 & -(x^3 + x^2) \\
\hline
 & \qquad x^2 + 0x \\
 & \qquad -(x^2 + x) \\
\hline
 & \qquad\quad -x - 5 \\
 & \qquad\quad -(-x - 1) \\
\hline
 & \qquad\qquad -4
\end{array}
$$

$$\frac{x^3+2x-5}{x+1} = x^2+x-1+\frac{-4}{x+1}$$

$$\Rightarrow \int x^2+x-1-\frac{4}{x+1}\,dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}-x-4\ln|x+1|+C$$

B) $\int \frac{x}{x+5}\,dx = \underbrace{\int 1 - \frac{5}{x+5}\,dx} = x - 5\ln|x+5| + C$

$$\begin{array}{r|l} & \ 1 \\ \hline x+5 & x + 0 \\ & -(x+5) \\ \hline & \quad -5 \end{array}$$

$$\int \frac{x}{x+5}\,dx = \int \frac{x+5-5}{x+5}\,dx = \underbrace{\int \frac{x+5}{x+5} - \frac{5}{x+5}\,dx}$$

C) $\int \frac{x^3+3x-5}{x^2+1}\,dx$

$$
\begin{array}{r|l}
 & x \\
x^2 + 0x + 1 & x^3 + 0x^2 + 3x - 5 \\
 & -(x^3 + 0x^2 + x) \\
\hline
 & 2x - 5
\end{array}
$$

C) $\int \frac{x^3+3x-5}{x^2+1}\,dx = \int x + \frac{2x-5}{x^2+1}\,dx = \int x + \frac{2x}{x^2+1} + \frac{-5}{x^2+1}\,dx$

$$= \int x\,dx + \underbrace{\int \frac{2x}{x^2+1}\,dx}_{u = x^2+1} - \int \frac{5}{x^2+1}\,dx$$

$$= \frac{x^2}{2} + \ln|x^2+1| - 5\arctan(x) + C$$

Example: Write out the partial fraction decomposition. Do not determine the numerical values of the coefficients.

A) $\frac{-3x+20}{x^3+3x^2-10x} = \frac{-3x+20}{x(x^2+3x-10)} = \frac{-3x+20}{x(x+5)(x-2)}$

$= \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-2}$

B) $\frac{x-3}{x(x+1)^3(x^2+5)} =$

$\frac{1}{x} + \frac{1}{x^2} = \frac{x}{x^2} + \frac{1}{x^2} = \frac{x+1}{x^2}$

$\frac{A}{x} + \frac{B}{(x+1)^3} + \frac{C}{(x+1)^2} + \frac{d}{x+1} + \frac{ex+f}{x^2+5}$

C) $\frac{x^2+2}{x^3(x^2-9)(x^2+16)^2}$

$(x-3)(x+3)$

$x^3 = (x+0)^3$

$= \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{d}{x-3} + \frac{e}{x+3} + \frac{fx+g}{(x^2+16)^2} + \frac{hx+i}{x^2+16}$

Example: Compute these integrals.

A) $\int \frac{-3x+20}{x^3+3x^2-10x}\,dx$

$$\frac{-3x+20}{x^3+3x^2-10x} = \frac{-3x+20}{x(x+5)(x-2)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-2}$$

$$\frac{-3x+20}{x(x+5)(x-2)} = \frac{A(x+5)(x-2)}{x(x+5)(x-2)} + \frac{Bx(x-2)}{x(x+5)(x-2)} + \frac{Cx(x+5)}{x(x+5)(x-2)}$$

$$\frac{-3x+20}{x(x+5)(x-2)} = \frac{A(x+5)(x-2) + Bx(x-2) + Cx(x+5)}{x(x+5)(x-2)}$$

$$-3x+20 = A(x+5)(x-2) + Bx(x-2) + Cx(x+5)$$

$$-3x+20 = A(x^2+3x-10) + B(x^2-2x) + C(x^2+5x)$$

$$0x^2 - 3x + 20 = Ax^2 + 3Ax - 10A + Bx^2 - 2Bx + Cx^2 + 5Cx$$

x^2) $0 = A + B + C$

x) $-3 = 3A - 2B + 5C$

Const) $20 = -10A \longrightarrow A = -2$

$0 = -2 + B + C$ $\qquad$ $-3 = -6 - 2B + 5C$

$2 = B + C$

$B = 2 - C$

$B = 1$

$3 = -2B + 5C$

$3 = -2(2-C) + 5C$

$3 = -4 + 2C + 5C$

$7 = 7C$

$1 = C$

$$= \int \frac{-2}{x} + \frac{1}{x+5} + \frac{1}{x-2}\, dx = -2\ln|x| + \ln|x+5| + \ln|x-2| + C$$

Page 6 method 2

Example: Compute these integrals.

A) $\int \frac{-3x+20}{x^3+3x^2-10x}\,dx$

$$\frac{-3x+20}{x^3+3x^2-10x} = \frac{-3x+20}{x(x+5)(x-2)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-2}$$

$$-3x+20 = A(x+5)(x-2) + Bx(x-2) + Cx(x+5)$$

if $x=0$)

$$20 = A(5)(-2) + 0 + 0$$

$$20 = -10A$$

$$-2 = A$$

$x=2$)

$$-6+20 = 14 = 0 + 0 + 2C\cdot 7$$

$$14 = 14C$$

$$1 = C$$

$x=-5$

$$15+20 = 0 + B(-5)(-7)+0$$

$$35 = 35B$$

$$1 = B$$

$$\int \frac{-2}{x} + \frac{1}{x+5} + \frac{1}{x-2}\,dx$$

$$= -2\ln|x| + \ln|x+5| + \ln|x-2| + C$$

B) $\int \frac{x+2}{x^3+2x}\,dx$

$$\frac{x+2}{x^3+2x} = \frac{x+2}{\underbrace{x(x^2+2)}} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

$$x+2 = A(x^2+2) + (Bx+C)x$$

$$x+2 = Ax^2 + 2A + Bx^2 + Cx$$

x^2 | $0 = A + B$

x | $1 = C$

Const | $2 = 2A \rightarrow A = 1 \rightarrow B = -1$

Short cut

$$\boxed{\begin{array}{l} \text{Let: } x = 0 \\ 2 = A(2) + 0 \\ A = 1 \end{array}}$$

$$= \int \frac{1}{x} + \frac{-x+1}{x^2+2}\,dx = \int \frac{1}{x} - \underbrace{\frac{x}{x^2+2}}_{u = x^2+2} + \underbrace{\frac{1}{x^2+2}}_{\text{from 7.3 work.}}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+2| + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C$$

C) $\displaystyle\int \frac{15x+5}{(x+2)^2(x^2+1)}\,dx$

$$\frac{15x+5}{(x+2)^2(x^2+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+d}{x^2+1}$$

$$15x+5 = A(x+2)(x^2+1) + B(x^2+1) + (Cx+d)(x+2)^2$$

$$= A(x^3+x+2x^2+2) + B(x^2+1) + (Cx+d)(x^2+4x+4)$$

$$15x+5 = Ax^3 + Ax + 2Ax^2 + 2A + Bx^2 + B + Cx^3 + 4Cx^2 + 4Cx + dx^2 + 4dx + 4d$$

x^3] $0 = A + C \longrightarrow \boxed{A = -C}$

x^2] $0 = 2A + B + 4C + d$

x] $15 = A + 4C + 4d$

Const] $5 = 2A + B + 4d$

Short cut. if $x = -2$

$-30+5 = B(4+1)$

$-25 = 5B$

$B = -5$

$0 = 2A - 5 - 4A + d$

$5 = -2A + d$

$\boxed{5 + 2A = d}$

$15 = A - 4A + 4d$

$15 = -3A + 4d$

$5 = 2A - 5 + 4d$

$10 = 2A + 4d$

$10 = 2A + 4(5+2A)$

$10 = 2A + 20 + 8A$

$-10 = 10A$

$-1 = A$

$A = -C$

$C = -A = 1$

$5 - 2 = d$

$d = 3$

C) $\int \frac{15x+5}{(x+2)^2(x^2+1)}\,dx = \int \frac{-1}{x+2} + \frac{-5}{(x+2)^2} + \underbrace{\frac{1x+3}{x^2+1}}\,dx$

$$= \int \frac{-1}{x+2}\,dx - \int \frac{5}{(x+2)^2}\,dx + \underbrace{\int \frac{x}{x^2+1}\,dx + \int \frac{3}{x^2+1}\,dx}$$

$u = x+2$

$u = x^2+1$

$$= -\ln|x+2| - \frac{-5}{x+2} + \frac{1}{2}\ln|x^2+1| + 3\arctan(x) + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$