Section 11.1: Sequences

Definition: A sequence is a list of numbers written in a definite order.
$a_1, a_2, a_3, \ldots$ or $\{a_n\}_{n=1}^{\infty}$

Example: Find a general formula for these sequences.

A) $\left\{\frac{5}{9}, \frac{6}{16}, \frac{7}{25}, \frac{8}{36}, \ldots\right\}$

$3^2 \; 4^2 \; 5^2 \; 6^2$

now shift the seq.

$$\boxed{j = n-2}$$

$j+2 = n$

$$\boxed{a_n = \frac{n+2}{n^2} \qquad n = 3, 4, 5, \ldots}$$

$$\boxed{j = 1, 2, 3, \ldots \qquad a_j = \frac{j+2+2}{(j+2)^2} = \frac{j+4}{(j+2)^2}}$$

$\longrightarrow a_n = \frac{n+4}{(n+2)^2} \qquad n = 1, 2, 3, \ldots$

Constant Rate of change

↳ a line

B) $\left\{\frac{3}{4}, \frac{6}{11}, \frac{9}{18}, \frac{12}{25}, \frac{15}{32}, \dots\right\}$

7 7 7 7

$$\boxed{a_n = \frac{3n}{7n-3} \qquad n = 1, 2, 3, \dots}$$

N value

(1, 4)

(2, 11)

$m = \frac{11-4}{2-1} = \frac{7}{1} = 7$

$y - 4 = 7(x-1)$

$y = 7x - 7 + 4$

$y = 7x - 3$

C) $\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$

Recursive seq.

$a_1 = 1$

$a_2 = 1$

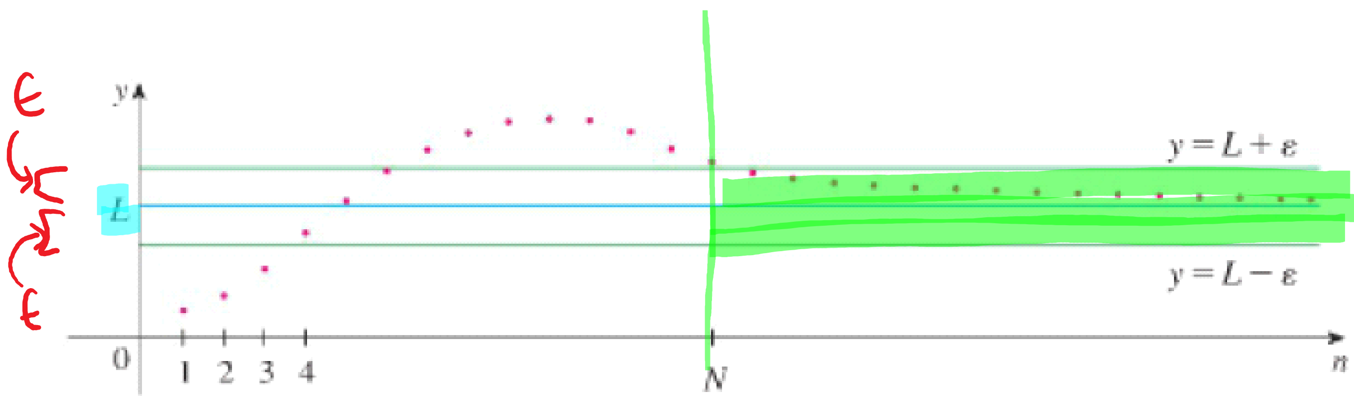
$a_n = a_{n-1} + a_{n-2}$ for $n = 3, 4, 5, \dots$

$a_3 = a_2 + a_1 = 1 + 1 = 2$

Page 3: Limit def. of converge

Definition: A sequence $\{a_n\}$ is said to have the limit L, written $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$, if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say that the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

Definition: If $\{a_n\}$ is a sequence, then $\lim_{n\to\infty} a_n = L$ means that for every $\epsilon > 0$ there is a corresponding integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.



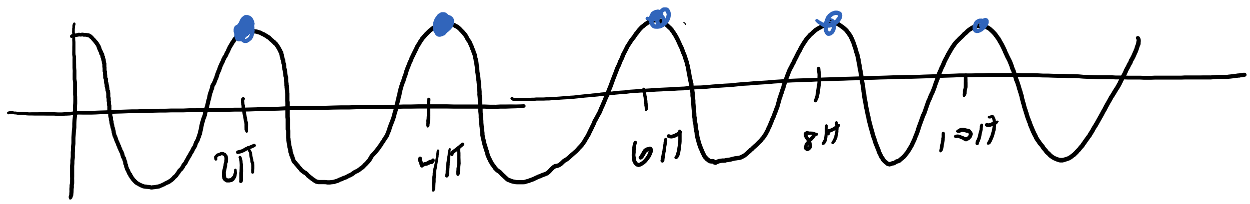
Example: Do these sequences converge or diverge.

A) $\{(-1)^n\}_{n=1}^{\infty}$ $= -1, 1, -1, 1, -1, 1, \dots\dots$

div.

B) $\{\cos(2n\pi)\}$ $= 1, 1, 1, 1, 1, 1, 1, \dots$

The seq will conv. to the # 1



C) $\left\{\frac{3n}{n+2}\right\}_{n=5}^{\infty}$

$$\lim_{n\to\infty} \frac{3n}{n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{3}{1+\frac{2}{n}} = \frac{3}{1+0} = 3$$

The seq. conv. to the # 3

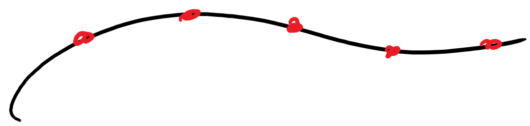
$$f(x) = \frac{3x}{x+2}$$

$$\lim_{x\to\infty} \frac{3x}{x+2} \overset{L'H}{=} \lim_{x\to\infty} \frac{3}{1}$$

$$= 3$$

f(x) is a function

THEOREM If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.



Example: Does the sequences converge or diverge? If it converges, give the value.

A) $\left\{\dfrac{n^2}{\ln(3+e^n)}\right\}$ $\qquad f(x) = \dfrac{x^2}{\ln(3+e^x)}$

$$\lim_{x\to\infty}\frac{x^2}{\ln(3+e^x)} \overset{L'H}{=} \lim_{x\to\infty}\frac{2x}{\frac{e^x}{3+e^x}} = \lim_{x\to\infty}\frac{2x(3+e^x)}{e^x}$$

$$= \lim_{x\to\infty}\frac{6x+2xe^x}{e^x}$$

$$= \lim_{x\to\infty}\frac{6x}{e^x}+\frac{2xe^x}{e^x} = \lim_{x\to\infty}\frac{6x}{e^x}+2x$$

$$= \infty$$

Thus the seq. div.

Seq. converges to 4

B) $\left\{ \frac{3n}{n+2} + \frac{n^2}{n^2+1} \right\}$

$$\lim_{n\to\infty} \left(\frac{3n}{n+2} + \frac{n^2}{n^2+1} \right) = 3 + 1 = 4$$

as a function of n

Limit Laws for Convergent Sequences: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n * \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \quad if \lim_{n\to\infty} b_n \neq 0$$

Squeeze Theorem for Sequences: If $a_n \leq b_n \leq c_n$ for $n \geq n_o$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

Example: Does the sequence a_n converge or diverge? If it converges, give the value.

A) $a_n = \frac{(-1)^n n^2}{n^2+1}$ $= (-1)^n b_n$ where $b_n = \frac{n^2}{n^2+1}$

as $n \to \infty$ $b_n \to 1$

$\lim_{x \to \infty} \frac{x^2}{x^2+1} \overset{L'H}{=} \cdots = 1$

means a_n diverge.

B) $a_n = \frac{(-1)^n 3n}{n^2+5} = (-1)^n b_n$ where $b_n = \frac{3n}{n^2+5}$

as $n \to \infty$ $b_n \to 0$

$\lim_{x\to\infty} \frac{3x}{x^2+5} \stackrel{L'H}{=} 0$

Thus $a_n \to 0$ as $n \to \infty$ i.e. a_n converges to zero

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Thm

$a_n = (-1)^n b_n$

if $b_n \to 0$ then $a_n \to 0$

if $b \to L \neq 0$ or b_n div. Then a_n div.

C) $a_n = \frac{n!}{n^n}$

$$\lim_{x \to \infty} f(x) \qquad f(x) = \frac{x!}{x^x}$$

$$b_n = 0 < \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots (n-1) \cdot n}{n \quad n \quad n \quad n \quad \cdots \quad n \quad n} < \frac{1}{n} = c_n$$

is 1

all of these are < 1

as $n \to \infty$ $\quad b_n \to 0 \qquad c_n \to 0$

by the Squeeze Thrm a_n converges to zero.

Example: Find the values of r so that $\{r^n\}$ converges. Determine what the series will converge to.

$$r^n \longrightarrow \begin{cases} 1 & \text{if } r = 1 & \text{conv.} \\ \infty & r > 1 & \text{div.} \\ \text{DNE} & r = -1 & \text{div.} \\ \text{DNE} & r < -1 & \text{div} \\ 0 & -1 < r < 1 & \text{conv.} \end{cases}$$

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is $a_1 < a_2 < a_3 < ....$ It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonoic** if it is either increasing or decreasing.

Example: Show that the sequence $a_n = \dfrac{n}{n^2+4}$ is a decreasing sequence.

$$f(x) = \frac{x}{x^2+4} \qquad f(n) = a_n$$

$$f'(x) = \frac{(x^2+4)(1) - x(2x)}{(x^2+4)^2} = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$$

$f'(x) = 0$ when $x = 2, -2$

f' -2 0 $+$ 1 2 $-$ 3

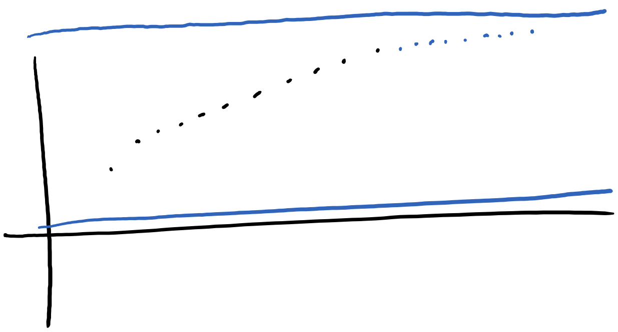
$f(x)$ is dec for $x \geq 2$

Thus a_n is dec for $n \geq 2$

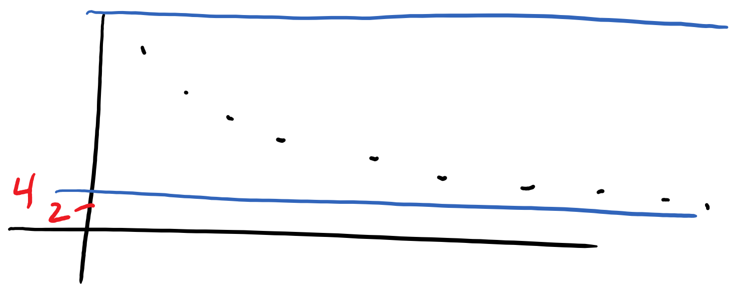
Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$. It is **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$. If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

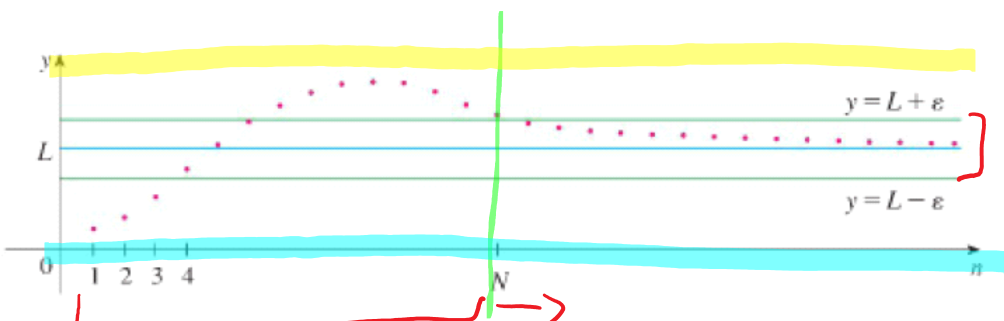
Inc.



dec



Question: If a sequence is convergent, is the sequence bounded? yes.



finite # of terms.

Example: You are given that the sequence given by $a_1 = \sqrt{5}$, $a_{n+1} = \sqrt{5 + a_n}$ is increasing and bounded above by 4.

Recursive.

$a_n = \sqrt{5 + a_{n-1}}$

Find $\lim_{n\to\infty} a_n$

seq converges.

$\lim_{n\to\infty} a_n = L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{5 + a_n}$$

$$L = \sqrt{5 + L}$$

$$L^2 = 5 + L$$

$$L^2 - L - 5 = 0$$

$$L = \frac{1 \pm \sqrt{1 - 4(1)(-5)}}{2}$$

$$L = \frac{1 \pm \sqrt{21}}{2}$$

$$L = \frac{1 + \sqrt{21}}{2}$$

$$L = \frac{1 - \sqrt{21}}{2}$$

negative

Thus the seq. conv. to $\frac{1 + \sqrt{21}}{2}$

Example: You are told that the sequence given by $a_1 = 1$, $a_{n+1} = 3a_n - 1$ is increasing. Does this sequence converge?

assume $\lim_{n \to \infty} a_n = L$ as $n \to \infty$ $a_n \to L$, $a_{n+1} \to L$

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} 3a_n - 1$$

$$L = 3L - 1$$

$$1 = 2L$$

$$L = \frac{1}{2}$$ ← doesn't make sense

our value does not make sense.

Thus we see by Inspection that $a_n \to \infty$

div.

Example: Assume that this sequence will converge. Give the exact value that it will converge to.

$a_1 = -1$ $\quad$ $a_{n+1} = \frac{1}{5}\left(a_n + \frac{44}{a_n}\right)$

$a_n \to L$ and $a_{n+1} \to L$ as $n \to \infty$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{5}\left(a_n + \frac{44}{a_n}\right)$$

$$L = \frac{1}{5}\left(L + \frac{44}{L}\right)$$

$$5L = L + \frac{44}{L}$$

$$4L = \frac{44}{L}$$

$$4L^2 = 44$$

$$L^2 = 11$$

$$L = \pm\sqrt{11}$$

$L = \sqrt{11}$ $\quad$ $L = -\sqrt{11}$

Conv. to $-\sqrt{11}$ since all terms are neg.