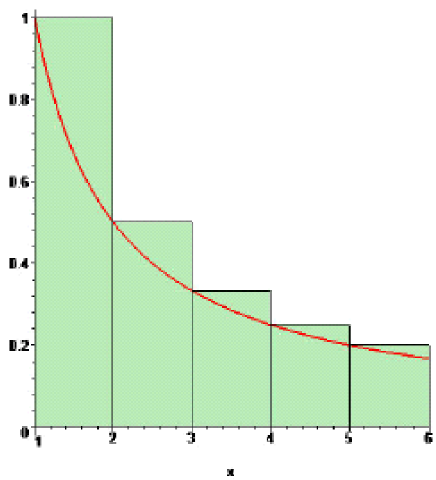
Note: In this section all series have positive terms.

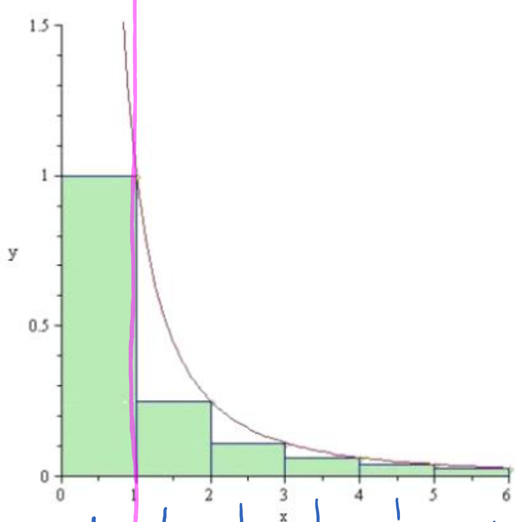
This graph was used to discuss the Harmonic series in section 11.2. We saw that

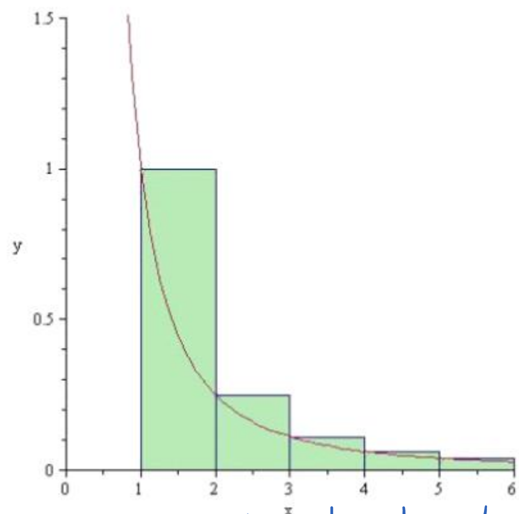
$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_{1}^{\infty} \frac{1}{x}\, dx > 0$$

and concluded that the Harmonic series diverged.



Here are the graphs for the function $f(x) = \frac{1}{x^2}$. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?





$$\underbrace{\sum_{n=2}^{\infty} \frac{1}{n^2} < \underbrace{\int_1^{\infty} \frac{1}{x^2}\,dx}}$$

p integrals conv. $p = 2$

$$\underbrace{\underbrace{\int_1^{\infty} \frac{1}{x^2}\,dx} < \sum_{n=1}^{\infty} \frac{1}{n^2}}$$

no conclusion.

Thus $\sum_{n=2}^{\infty} \frac{1}{n^2}$ will converge

Thus $\sum_{n=1}^{\infty} \frac{1}{n^2}$ will conv.

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^p}}_{p\text{-series}} \begin{cases} p > 1 & \text{the series will converge} \\ p \le 1 & \text{the series will diverge} \end{cases}$$

p-series L

The Integral Test Suppose f is continuous, positive, decreasing function on $[1, \infty)$, or on $[A, \infty)$, and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. In other words:

(a) If $\int_1^{\infty} f(x)\, dx$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\int_1^{\infty} f(x)\, dx$ is divergent then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example: Assume that $\sum_{n=1}^{\infty} a_n$ is a series where $f(n) = a_n$ and $\int_1^{\infty} f(x)\, dx$ converges to the number L. Does this mean that the series $\sum_{n=1}^{\infty} a_n$ converge to L?

$$\int_1^{\infty} f(x)\, dx = L$$

no.

$$\int_1^{\infty} \frac{1}{x^2}\, dx = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Example: Do these series converge or diverge?

A) $\sum_{n=1}^{\infty} \frac{n^2}{1+n^2}$

$$\lim_{n \to \infty} \frac{n^2}{1+n^2} \overset{L'H}{=} 1$$

by the test for div.
This series will diverge.

B) $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

$\lim_{n \to \infty} \frac{1}{1+n^2} = 0$

may or may not conv.

$f(x) = \frac{1}{1+x^2}$

continuous ✓
positive ✓
dec. ✓

$f'(x) = \frac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2}$

$f'(x) = \frac{-2x}{(1+x^2)^2} < 0$ for $x > 0$

Dec ✓

$$\int_1^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{1+x^2}\,dx = \lim_{t \to \infty} \arctan(x)\Big|_1^t$$

$$= \lim_{t \to \infty} \left[\arctan(t) - \arctan(1)\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} \checkmark$$

The integral converges.

by the The integral test $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ will also conv.

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method 2)

$\int_1^{\infty} \frac{1}{1+x^2}\,dx$

$1+x^2 > x^2$

$\frac{1}{1+x^2} < \frac{1}{x^2}$

Know $\int_1^{\infty} \frac{1}{x^2}\,dx$ conv. p-integral $p = 2$

Know $\int_1 \frac{1}{x^2} dx$ conv. $p = 2$

Thus $\int_1^\infty \frac{1}{1+x^2} dx$ will conv. by the comparison thrm

Thus $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ will conv. by the Integral test.

C) $\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n}$

$f(x) = \frac{(\ln(x))^2}{x}$

cont. ✓ for $x > 0$

positive ✓

dec. ✓

$$f'(x) = \frac{x \cdot 2\ln(x) \cdot \frac{1}{x} - (\ln(x))^2 \cdot 1}{x^2}$$

$$= \frac{2\ln(x) - (\ln(x))^2}{x^2}$$

$$= \frac{\ln(x)\left[2 - \ln(x)\right]}{x^2}$$

$f'(x) < 0$ if $\ln(x) > 2$, $x > e^2$

$$\int_1^\infty \frac{(\ln(x))^2}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{(\ln(x))^2}{x}\,dx$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \int_{x=1}^{x=t} u^2\,du = \lim_{t\to\infty} \left.\frac{u^3}{3}\right|_{x=1}^{x=t}$$

$$= \lim_{t\to\infty} \left.\frac{(\ln(x))^3}{3}\right|_1^t$$

$$= \lim_{t\to\infty} \frac{(\ln(t))^3}{3} - 0 = \infty$$

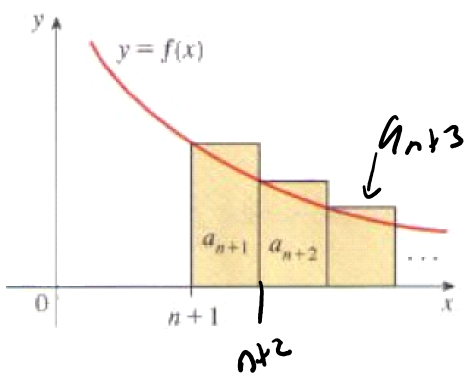
The improper integral diverges

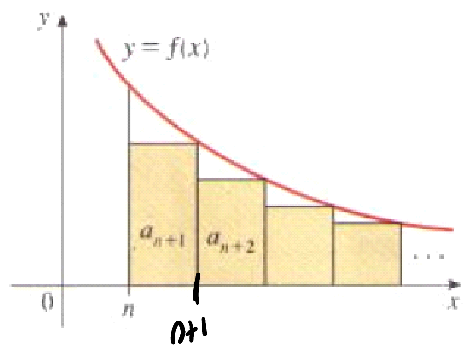
by the Integral test the series will div.

Remainder Estimate For The Integral Test Suppose that $\sum a_n$ converges by the Integral Test to the number s. If s_n is a partial sum that approximates this series, define R_n to be the remainder, i.e. $s_n + R_n = s$, then we get the following bounds on the remainder and the sum:

$S_n + R_n = S$

$$\sum_{i=1}^{\infty} a_i = \underbrace{a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n}_{S_n} + \underbrace{a_{n+1} + a_{n+2} + \ldots}_{R_n} = s$$





$$\boxed{\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx}$$

$S_n + R_n = S$

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{n}^{\infty} f(x)\,dx$$

Example: Given that this series converges by the Integral test. $\sum_{i=1}^{\infty} \frac{30}{i^4}$.

$f(x) = \frac{30}{x^4}$

A) Find the bounds on R_{10}
B) What bounds are on the sum for this series?
C) If you wanted the error to be less than 0.005, what is the smallest value of n should you use?

upper bound

$$R_{10} \le \int_{10}^{\infty} \frac{30}{x^4}\,dx = \lim_{t\to\infty} \int_{10}^{t} 30x^{-4}\,dx$$

$$= \lim_{t\to\infty} \left.\frac{30x^{-3}}{-3}\right|_{10}^{t} = \lim_{t\to\infty} \left.\frac{-10}{x^3}\right|_{10}^{t}$$

$$= \lim_{t\to\infty} \left(\frac{-10}{t^3} - \frac{-10}{10^3}\right) = +\frac{1}{10^2} = \frac{1}{100}$$

Lower bound

$$R_{10} \ge \int_{11}^{\infty} \frac{30}{x^4}\,dx = \cdots = 0 + \frac{+10}{11^3} = \frac{10}{11^3}$$

A

$$\frac{10}{11^3} \le R_{10} \le \frac{1}{100}$$

Bound on S.

need $S_{10} = 1 + \frac{1}{2^4} + \cdots + \frac{1}{10^4} = 32.4610575$

$10 \quad S \le 32.4610575 + \frac{1}{100}$

$$32.4610975 + \frac{10}{11^3} \leq S \leq 32.4610975 + \frac{1}{100}$$

Page 8 pt b

Example: Given that this series converges by the Integral test. $\sum_{i=1}^{\infty} \frac{30}{i^4}$.

A) Find the bounds on R_{10}
B) What bounds are on the sum for this series?
C) If you wanted the error to be less than 0.005, what is the smallest value of n should you use?

$$R_n < .005$$

$$R_n \le \int_n^{\infty} \frac{30}{x^4}\,dx < .005$$

$$\int_n^{\infty} \frac{30}{x^4}\,dx = \lim_{t\to\infty} \int_n^{t} \frac{30}{x^4}\,dx = \ldots = \frac{10}{n^3}$$

$$\frac{10}{n^3} < .005$$

$$\frac{10}{.005} < n^3$$

$$2000 < n^3$$

$$\sqrt[3]{2000} < n \rightarrow n > 12.599$$

$$n = 13$$

Example: The following series converges by integral test. Determine the smallest number of terms that should be used if the maximum error of the partial sum should be less than 0.05?

$$\sum_{i=1}^{\infty} \frac{1}{i^3}.$$

$$R_n \le \int_n^{\infty} \frac{1}{x^3}\,dx < .05 = \frac{1}{20}$$

$$\int_n^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_n^t x^{-3}\,dx = \lim_{t\to\infty} \left.\frac{x^{-2}}{-2}\right|_n^t$$

$$= \lim_{t\to\infty} \underbrace{\frac{-1}{2t^2}} - \frac{-1}{2n^2} = \frac{1}{2n^2}$$

$$\frac{1}{2n^2} < \frac{1}{20} \longrightarrow 20 < 2n^2$$

$$10 < n^2$$

$$\sqrt{10} < n \longrightarrow n = 4$$