Pg 1: Comparison Test

## Section 11.4: The Comparison Tests

**Note: In this section all series have positive terms.**

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**The Comparison Test (Strict Comparison):** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

we are testing

(a) If $\sum b_n$ is convergent and $a_n \le b_n$ for all $n$, then $\sum a_n$ is also convergent.

(b) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all $n$, then $\sum a_n$ is also divergent.

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Known series

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Example: Do these series converge or diverge?

A) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{5n^3 + n^2 + 1}$

$5n^3 + n^2 + 1 > 5n^3$

$\dfrac{1}{5n^3 + n^2 + 1} < \dfrac{1}{5n^3}$

$\dfrac{6}{5n^3 + n^2 + 1} < \dfrac{6}{5n^3}$

$\sum \dfrac{6}{5n^3} = \dfrac{6}{5} \underbrace{\sum \dfrac{1}{n^3}}$

p-series $p = 3 > 1$

This series converges

by the comparison test $\sum \dfrac{6}{5n^3 + n^2 + 1}$

will also converge.

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B) $\sum_{n=1}^{\infty} \frac{3^{2n+1}}{7^n+5}$

$7^n+5 > 7^n$

$\frac{1}{7^n+5} < \frac{1}{7^n}$

$\frac{3^{2n+1}}{7^n+5} < \frac{3^{2n+1}}{7^n}$

geometric

$\sum_{n=1} \frac{3^{2n+1}}{7^n}$

$= \frac{3^3}{7^1} + \frac{3^5}{7^2} + \frac{3^7}{7^3} + \cdots$

$r = \frac{3^2}{7} = \frac{9}{7}$

$|r| > 1$ div.

our comparison did not help.

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Test for div.

$$\lim_{n\to\infty} \frac{3^{2n+1}}{7^n+5} \overset{L'H}{=} \lim_{n\to\infty} \frac{2\cdot 3^{2n+1}\ln(3)}{7^n\ln(7)}$$

$$= \lim_{n\to\infty} \frac{2\cdot\ln(3)}{\ln(7)} \cdot \frac{3^1\cdot 3^{2n}}{7^n}$$

$$= \lim_{n\to\infty} \frac{2\cdot 3\cdot\ln(3)}{\ln(7)} \cdot \boxed{\frac{9^n}{7^n}} = \infty$$

$n$

Series div. by the test for

$$\lim_{n\to\infty}\left(\frac{9}{7}\right)^n = \infty$$

by The test for div.

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$a_n$

C) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{5^n - 2}}$

$5^n - 2 < 5^n$

$\dfrac{1}{5^n - 2} > \dfrac{1}{5^n}$

$\lim_{n \to \infty} a_n = 0$

$\sum \dfrac{1}{5^n}$ geometric

$r = \dfrac{1}{5}$

$|r| < 1$ conv.

no conclusion given by the comparison test.

# Pg 5: Limit comparison test

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**Limit Comparison Test(LCT):** Suppose that $\underbrace{\sum a_n}_{\text{Testing}}$ and $\underbrace{\sum b_n}_{\text{Known}}$ are series with positive terms and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \geq 0$$

If $L > 0$ then both series converge or both series diverge.

If $L = 0$ and $\sum b_n$ converge, then $\sum a_n$ converge.

If $L = \infty$ and $\sum b_n$ diverge, then $\sum a_n$ diverge.

(Note: This test is slightly different that the test given in the book.)

Example: Do these series converge or diverge?

A) $\sum_{n=1}^{\infty} \frac{1}{5^n - 2}$

Looks like $\sum_{n=1}^{\infty} \frac{1}{5^n}$

geometric $r = \frac{1}{5}$

$|r| < 1$ conv.

$$\lim_{n \to \infty} \frac{\frac{1}{5^n - 2}}{\frac{1}{5^n}} = \lim_{n \to \infty} \frac{5^n}{5^n - 2} \stackrel{L'H}{=} \lim_{n \to \infty} \frac{5^n \ln 5}{5^n \ln(5)}$$

$$= 1 > 0$$

LCT says <u>both</u> series have the same div/conv characteristic.

Thus our series $\sum_{n=1}^{\infty} \frac{1}{5^n - 2}$ will converge.

B) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt{n^2+2n}-7}$

use LCT with

$$\sum_{n=1} \frac{1}{\sqrt{n^2}} = \sum_{n=1} \frac{1}{n}$$

p-series $p=1$ div.

$$\lim_{n\to\infty} \frac{\dfrac{5}{\sqrt{n^2+2n}-7}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{5n}{\sqrt{n^2+2n}-7}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{5}{\frac{1}{2}(2n+2)(n^2+2n)^{-1/2}} = \lim_{n\to\infty} \frac{5\sqrt{n^2+2n}}{n+1}$$

$$= \lim_{n\to\infty} \frac{5\sqrt{n^2(1+\frac{2}{n})}}{n+1} = \lim_{n\to\infty} \frac{5n\sqrt{1+\frac{2}{n}}}{n+1} \quad \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{5\sqrt{1+\frac{2}{n}}}{1+\frac{1}{n}} = \frac{5\sqrt{1+0}}{1+0} = 5$$

LCT says both series do the same thing.

Thus the series will div.

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C) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Test for div.: may or may not conv

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n)}{n^3} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{3n^2}{1}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{3n^2} = \frac{1}{3n^3} = 0$$

LCT with $\sum \frac{1}{n^3}$ → conv. p-series $p=3$

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \ln(n) = \infty$$

no conclusion

LCT with $\sum \frac{1}{n}$ → div. p-series $p=1$

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln(n)}{n^3} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\ln(n)}{n^2} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2n}{1}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{2n} = \lim_{n\to\infty} \frac{1}{2n^2} = 0$$

no conclusion

LCT $\sum \frac{1}{n^2}$ p-series $p=2$ conv.

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln(n)}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

by our LCT the series will conv.

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D) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+5n}{2^n(n^2+1)}$

LCT with $\displaystyle\sum_{n=1} \frac{1}{2^n}$ geometric $r = \frac{1}{2}$ conv.

$$\lim_{n\to\infty} \frac{\frac{3n^2+5n}{2^n(n^2+1)}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{3n^2+5n}{\cancel{2^n}(n^2+1)} \cdot \frac{\cancel{2^n}}{1}$$

$$= \lim_{n\to\infty} \frac{3n^2+5n}{n^2+1} = 3$$

LCT says Both do the same.

Thus our series will conv.

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E) $\sum_{n=2}^{\infty} \frac{5+cos(n)}{\sqrt{n-1}} = J$

$-1 \le \cos(n) \le 1$

$4 \le 5+\cos(n) \le 6$

$\underbrace{\frac{4}{\sqrt{n-1}}}_{\text{positive}} \le \frac{5+\cos(n)}{\sqrt{n-1}} \le \underbrace{\frac{6}{\sqrt{n-1}}}_{\text{positive}}$

examine $\sum \frac{4}{\sqrt{n-1}}$

use LCT with $\sum \frac{1}{\sqrt{n}}$ p-series $p=\frac{1}{2}$ div.

$$\lim_{n\to\infty} \frac{\frac{4}{\sqrt{n-1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{4\sqrt{n}}{\sqrt{n-1}} = \lim_{n\to\infty} 4\sqrt{\frac{n}{n-1}}$$

$$= 4\sqrt{1} = 4$$

Thus by LCT $\sum \frac{4}{\sqrt{n-1}}$ will div.

by comparison test the series J will also div.