Section 6.4: Work

The work on an object(moving in a straight line) is defined to be the product of the force on the object and the distance that the object moves. W = Fd.

In the SI metric system force is measured in newtons(1N = 1 kg*m/s^2). If displacement is measured in meters then work has the units of N*m, which is also called a joule (J). Note that force can be viewed as the product of the mass of an object(measured in kg) and it acceleration(measured in m/s^2) , i.e. F = ma.
In the British system the fundamental unit of force is the pound.

Example: How much work is done lifting a 3kg object from the floor to a distance of 2 m in the air?

$F = ma = 3\,kg\,(9.8\,m/s^2)$
$= 29.4\,N$

$W = Fd = 29.4(2)$
$= 58.8\,Nm = 58.8\,J$

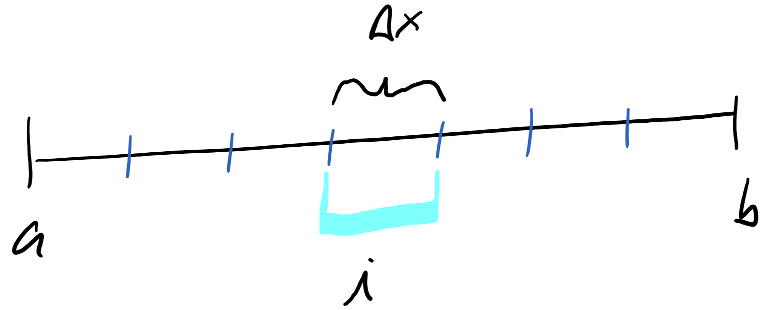
Example: How much work is done lifting a 50lb weight 3 ft off the ground?

$F = 50\,lb \qquad d = 3\,ft$

$W = F \cdot d = 50(3)$
$= 150\,ft \cdot lbs.$

Variable Force:

$f(x)$ = non constant force



$$W \approx \sum f(x_i)\Delta x \longrightarrow W = \int_a^b f(x)\,dx$$

Example: An object is moved along the x-axis by a force $F = 3x^2 + 1$, in Newtons, when the object is x meters from the origin.

A) Find the work done moving the object from the origin to a distance of 10m to the right of the origin.

$$W = \int_0^{10} 3x^2+1 \, dx = 1010 \, Nm = 1010 \, J$$

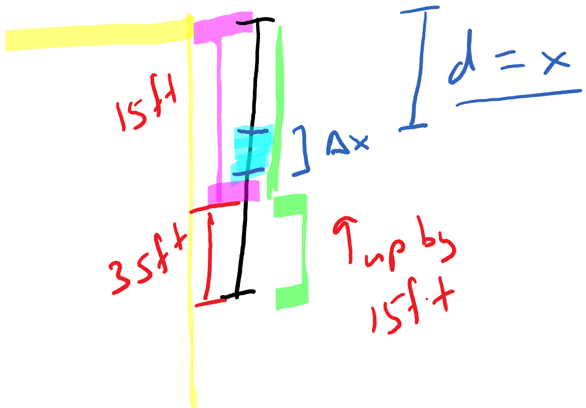
B) Find the work done moving the object from 5 meters from the origin to 12 meters from the origin.

$$W = \int_5^{12} 3x^2+1 \, dx = 1610 \, J$$

Example: A 100 pound cable is 50 feet long and hangs over the edge of a tall building. How much work is required to pull 15 feet of the cable to the top of the building?

cable

$$\frac{100\,lb}{50\,ft} = 2\ lb/ft$$



$$W = F \cdot d$$

$$F = 2\ lb/ft \cdot \Delta x\ ft$$

$$F = 2\Delta x\ \ lbs$$

$$W = 2\Delta x \cdot x = 2x\,\Delta x$$

first 15ft

$$W = \int_0^{15} 2x\,dx = 225\ ft\cdot lbs$$

last 35ft

$$35ft \cdot 2\,lb/ft = 70\ lbs$$

$$\ast\ 15\,ft$$

$$W = 1050\ ft\cdot lbs$$

$$\text{Answer} = 225 + 1050 = 1275\ ft\cdot lbs$$

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Example: A 100 pound cable is 50 feet long and hangs over the edge of a tall building. How much work is required to pull 15 feet of the cable to the top of the building?

Cable: 2 lbs/ft

pull up

$x = 0$ $\quad$ $F = 100 \text{ lbs} = 50 \text{ ft} \cdot 2 \text{ lbs/ft}$

$x = 50$ $\quad$ $F = 0 \text{ lbs.}$

$F = 100 - 2x$ $\quad$ non constant force.

$$W = \int_0^{15} 100 - 2x \, dx = \dots = 1275 \text{ ft} \cdot \text{lbs.}$$

method 2

Example: A cable that weighs 4lb/ft is used to lift 700 lb of coal up a mine shaft 300ft deep. Find the work done.

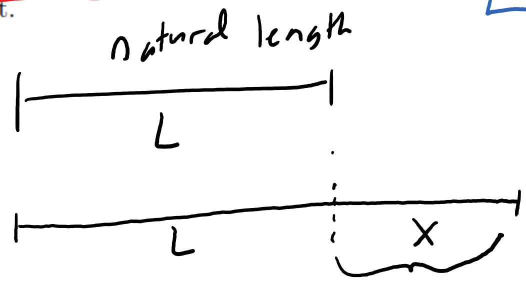
x = # of ft of cable pulled up.

$$F = \underbrace{300(4) - 4x}_{\text{cable}} + \underbrace{700}_{\text{coal}}$$

$$F = 1900 - 4x$$

$$W = \int_0^{300} 1900 - 4x \, dx = 390000 \text{ ft} \cdot \text{lbs.}$$

Hooke's Law states that the force required to maintain a spring stretched x units beyond it natural length is proportional to x, $f(x) = kx$, where k is a positive constant.



Example: A force of 9lb is required to stretch a spring from its natural length of 6in to a length of 8 in. Find the work done in stretching the spring from its natural length to a length of 10 in.

$X = 8 - 6 = 2\text{in}$

find work when

$X = 10 - 6 = 4\text{in}$

$f(x) = Kx$

$9 = K(2)$

$4.5 = K$

$f(x) = 4.5x$

$$W = \int_0^4 4.5x\,dx = \frac{4.5x^2}{2}\Big|_0^4$$

$= 36$ inch · pound

$f(x) = Kx$

$9 = K\left(\frac{2}{12}\right)$

$\frac{12(9)}{2} = K$

$K = 54$

$$W = \int_0^{4/12} 54x\,dx = \cdots = 3 \text{ ft}\cdot\text{lbs}$$

Example: A spring has a natural length of 6 inches. It takes 12 ft lbs of work to stretch the spring from 6 inches to 8 inches. Find the work it takes to stretch the spring from 8 inches to 9 inches.

$$12 \text{ ft}\cdot\text{lbs} = \int_0^{2/12} kx\,dx$$

$$12 = \frac{kx^2}{2}\Big|_0^{2/12}$$

$$12 = \frac{k}{2}\left(\frac{2}{12}\right)^2 = \frac{4k}{2(144)} = \frac{2k}{144} = \frac{k}{72}$$

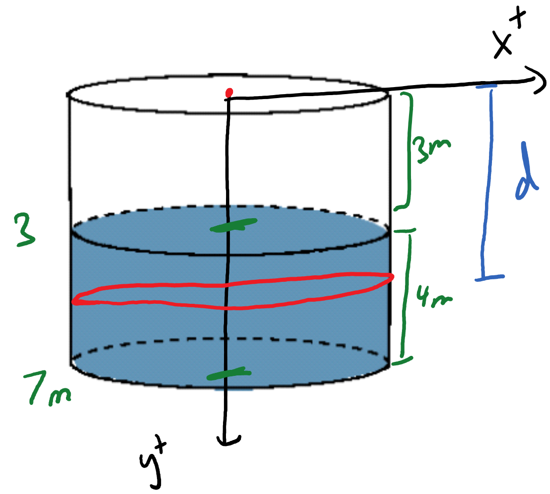
$$12 = \frac{k}{72}$$

$$k = 72(12) = 864$$

$$f(x) = 864x$$

$$W = \int_{2/12}^{3/12} 864x\,dx = 15 \text{ ft}\cdot\text{lbs}$$

Example: A tank is in the shape of an upright cylinder with a radius of 3m and a height of 7m. The tank is full of water to a depth of 4m. Find the work required to pump the water to the top of the tank.



$d = y$

$W = F \cdot d$

$F = mass \cdot a$
$= m \cdot g$

mass = density · Volume
$= \rho V$
rho

$F = \rho V g$
$F = \rho g V$

ρg weight density

find the Volume.

$V = \pi r^2 \Delta y$
$= \pi x^2 \Delta y$
$= \pi 3^2 \Delta y$
$V = 9\pi \Delta y$

for H_2O

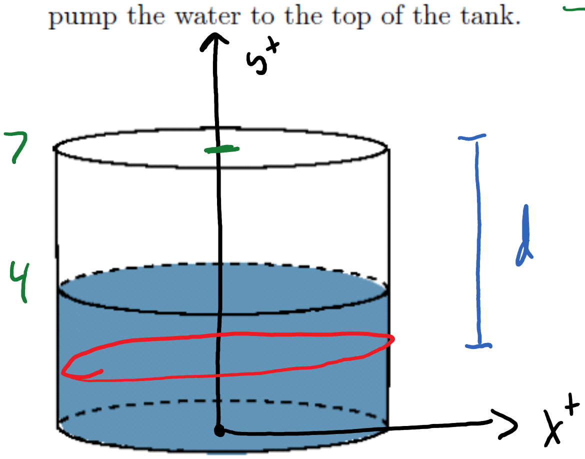
$\rho g = \frac{1000\ kg}{m^3} \cdot \frac{9.8m}{s^2}$
$= 9800\ \frac{kg\ m}{s^2} / m^3$
$\rho g = 9800\ N/m^3$

$F = 9800 \cdot 9\pi \Delta y$

$$W = \int_3^7 9800 \cdot 9\pi \cdot y\ dy = 176400\pi\ J$$

Example: A tank is in the shape of an upright cylinder with a radius of 3m and a height of 7m. The tank is full of water to a depth of 4m. Find the work required to pump the water to the top of the tank.

$\rho g = 9800\ N/m^3$



$d = 7 - y$

$F = \rho g V = \rho g 9\pi \Delta y$

$V = \pi r^2 \Delta y = \pi (3)^2 \Delta y$
$= 9\pi \Delta y$

$$W = \int_0^4 \rho g\, 9\pi (7-y)\, dy$$

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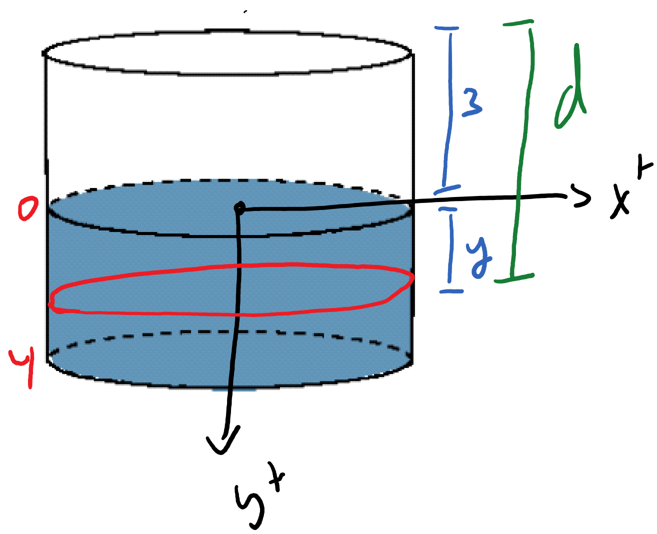


$d = y + 3 = y - (-3)$

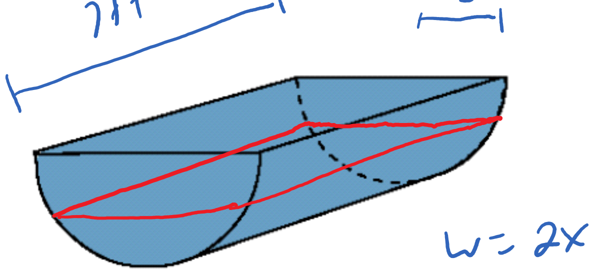
$F = \rho g V$

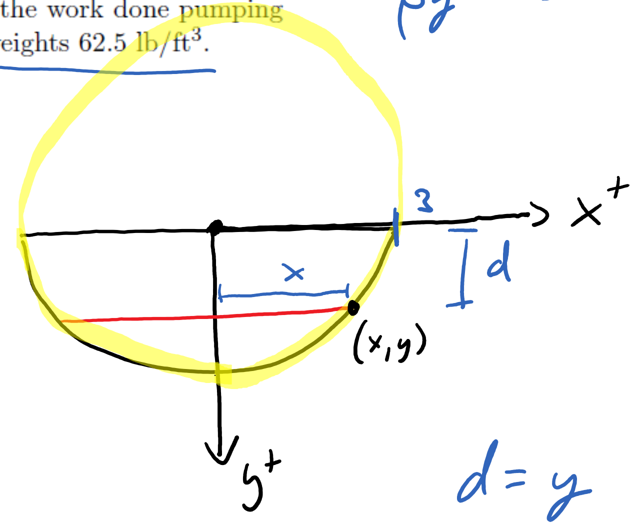
$V = \pi r^2 \Delta y = 9\pi \Delta y$

$$W = \int_0^4 \rho g \cdot 9\pi (y+3)\, dy$$

Example: A trough is in the shape of a half cylinder(on its side). The length of the trough is 7ft and it has a diameter of 6 ft. Assuming that the trough if full of water. Set up the integral that would be used to compute the work done pumping the water to the top of the tank. Use the fact that water weights 62.5 lb/ft^3.

$\rho g = 62.5 \text{ lb/ft}^3$





$d = y$

$V = lw\,\Delta y$

$= 7(2x)\,\Delta y$

$V = 14\cancel{x}\,\Delta y$

$V = 14\sqrt{9-y^2}\,\Delta y$

$x^2 + y^2 = r^2$

$x^2 + y^2 = 9$

$x = \pm\sqrt{9-y^2}$

$$W = \int_0^3 \rho g \cdot 14y\sqrt{9-y^2}\,dy$$

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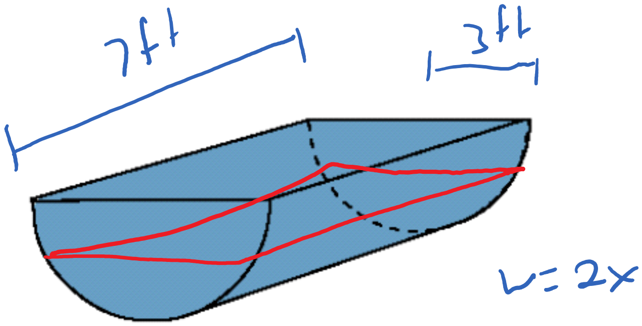
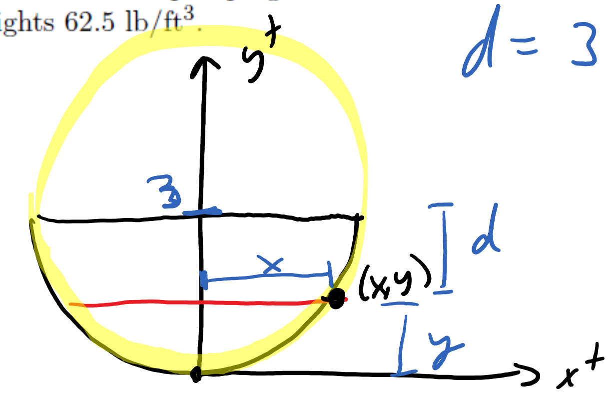




$d = 3 - y$

$V = lw\,\Delta y$

$V = 7(2x)\,\Delta y$

$V = 14x\,\Delta y$

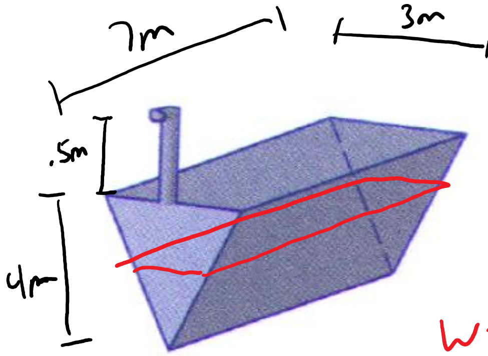
$(x-h)^2 + (y-k)^2 = r^2$

$x^2 + (y-3)^2 = 3^2$

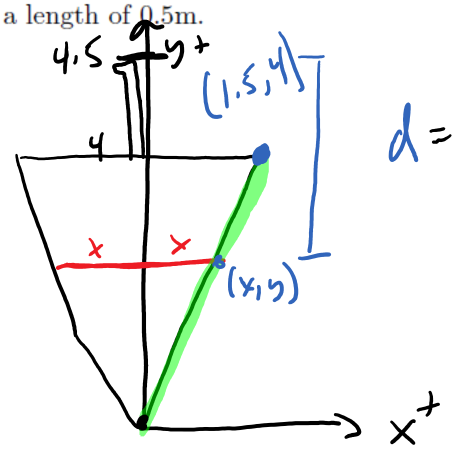
$x = \pm\sqrt{9-(y-3)^2}$

$$W = \int_0^3 \rho g\,(3-y)\cdot 14\sqrt{9-(y-3)^2}\;dy$$

Example: A triangular trough, with an isosceles triangle for an end, has a length of 7m, a distance of 3m across the top and a height of 4m. Assume that the tank is full of water. Set up the integrals that would give the work done pumping the water through a spout located at the top of the trough with a length of 0.5m.



$w = 2x$



$d = 4.5 - y$

$V = l w \Delta y$

$V = 7 w \Delta y = 14 x \Delta y$

$V = 14\left(\frac{3}{8} y\right) \Delta y$

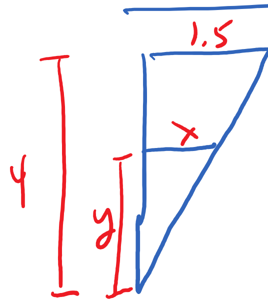
$(0,0) \quad (1.5,4)$

$m = \frac{4-0}{1.5-0}$

$m = \frac{4}{\frac{3}{2}} = \frac{8}{3}$

$y = \frac{8}{3} x$

$x = \frac{3}{8} y$



by similar triangle

$\frac{x}{y} = \frac{1.5}{4}$

$x = \frac{1.5}{4} y = \frac{3}{2} \cdot \frac{1}{4} y$

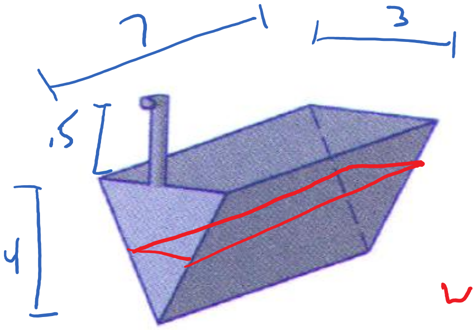
$x = \frac{3}{8} y$

$\rho g = 9800 \ N/m^3$

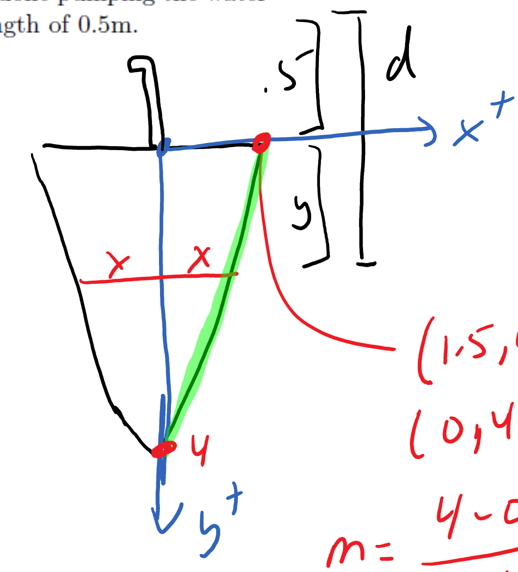
$$w = \int_0^4 \rho g \, 14 \cdot \frac{3}{8} y \cdot (4.5 - y) \, dy$$

Example: A triangular trough, with an isosceles triangle for an end, has a length of 7m, a distance of 3m across the top and a height of 4m. Assume that the tank is full of water. Set up the integrals that would give the work done pumping the water through a spout located at the top of the trough with a length of 0.5m.

$d = y + .5$



$w = 2x$



$(1.5, 0)$

$(0, 4)$

$$m = \frac{4-0}{0-1.5} = \frac{4}{\frac{-3}{2}}$$

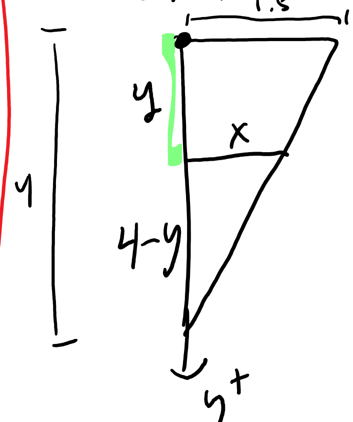
$$m = -\frac{8}{3}$$

$$V = lw\,\Delta y$$
$$= 7(2x)\,\Delta y$$
$$= 14x\,\Delta y$$

$$V = 14\left(-\frac{3}{8}(y-4)\right)\Delta y$$

$$y - 4 = -\frac{8}{3}(x-0)$$
$$y - 4 = -\frac{8}{3}x$$
$$-\frac{3}{8}(y-4) = x$$

Similar triangles



$$\frac{x}{4-y} = \frac{1.5}{4} = \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}$$

$$x = \frac{3}{8}(4-y)$$

$$W = \int^{4} \rho g\; 14\left(-\frac{3}{8}(y-4)\right)(y+.5)\,dy$$

$$W = \int_0^4 \rho g \; 14\left(-\frac{3}{8}(y-4)\right)(y+.5)\; dy$$