Section 7.2: Trig Integrals

Extremly Useful Trig Identities

$$\boxed{\begin{aligned}\sin^2(x) &= \tfrac{1}{2}(1-\cos(2x))\\ \cos^2(x) &= \tfrac{1}{2}(1+\cos(2x))\end{aligned}}$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\tan^2(x) + 1 = \sec^2(x)$$

$$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

Sometimes Useful Trig Identities

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

Compute these integrals

Example: $\int \sin(5x)\sin(4x)\ dx$ $= \int \frac{1}{2}\left[\cos(5x-4x) - \cos(5x+4x)\right] dx$

$$= \frac{1}{2}\int \cos(x) - \cos(9x)\ dx$$

$$= \frac{1}{2}\left[\sin(x) - \frac{1}{9}\sin(9x)\right] + C$$

Example: $\int \sin(7x)\cos^4(7x)\,dx$ $= \int -\frac{1}{7} u^4 \, du$

$u = \cos(7x)$

$du = -7\sin(7x)\,dx$

$-\frac{1}{7}du = \sin(7x)\,dx$

$= -\frac{1}{7}\frac{u^5}{5} + C$

$= -\frac{1}{35}\cos^5(7x) + C$

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$$\int \sin^m(x)\cos(x)\,dx$$

$u = \sin(x)$

$$\int \sec^m(x)\sec(x)\tan(x)\,dx$$

$u = \sec(x)$

$$\int \cos^m(x)\sin(x)\,dx$$

$u = \cos(x)$

$$\int \cot^m(x)\csc^2(x)\,dx$$

$u = \cot(x)$

$$\int \tan^m(x)\sec^2(x)\,dx$$

$u = \tan(x)$

$$\int \csc^m(x)\csc(x)\cot(x)\,dx$$

$u = \csc(x)$

Example: $\int \sin^2(3x)\, dx$

$$= \int \frac{1}{2}\left(1 - \cos(6x)\right)\, dx$$

$$= \frac{1}{2}\int 1 - \cos(6x)\, dx$$

$$= \frac{1}{2}\left[x - \frac{1}{6}\sin(6x)\right] + C$$

$$= \frac{x}{2} - \frac{x}{12}\sin(6x) + C$$

Example: $\int \cos^4(x)\,dx = \int \cos^2(x) \cdot \cos^2(x)\,dx$

$$= \int \frac{1}{2}\left(1+\cos(2x)\right) \cdot \frac{1}{2}\left(1+\cos(2x)\right) dx$$

$$= \frac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)\quad dx$$

$$= \frac{1}{4}\int 1 + 2\cos(2x) + \frac{1}{2}\left[1+\cos(4x)\right]\quad dx$$

$$= \frac{1}{4}\int 1 + 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x)\quad dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \frac{2}{2}\sin(2x) + \frac{1}{2}\cdot\frac{1}{4}\sin(4x)\right] + C$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \sin(2x) + \frac{1}{8}\sin(4x)\right] + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

$\sin^2\theta + \cos^2\theta = 1 \rightarrow \sin^2\theta = 1 - \cos^2\theta$

Example: $\int \sin^5(2x)\, dx = \int \sin^4(2x) \cdot \sin(2x)\, dx$

$$= \int \sin^2(2x)\sin^2(2x)\, \sin(2x)\, dx$$

$$= \int (1-\cos^2(2x))(1-\cos^2(2x))\, \sin(2x)\, dx$$

$u = \cos(2x)$

$du = -2\sin(2x)\, dx$

$-\frac{1}{2}\, du = \sin(2x)\, dx$

$$= \int -\frac{1}{2}(1-u^2)(1-u^2)\, du$$

$$= -\frac{1}{2}\int 1 - 2u^2 + u^4\, du$$

$$= -\frac{1}{2}\left[u - \frac{2u^3}{3} - \frac{u^5}{5}\right] + C$$

$$= -\frac{1}{2}\cos(2x) + \frac{1}{3}\cos^3(2x) + \frac{1}{10}\cos^5(2x) + C$$

Example: $\int \sin^4(3x)\cos^3(3x)\,dx = \int \sin^4(3x)\cos^2(3x)\cos(3x)\,dx$

$$= \int \sin^4(3x)\left(1-\sin^2(3x)\right)\cos(3x)\,dx$$

$u = \sin(3x)$

$du = 3\cos(3x)\,dx$

$\frac{1}{3}du = 3\cos(x)\,dx$

$$= \int u^4(1-u^2)\,\frac{1}{3}\,du$$

$$= \frac{1}{3}\int u^4 - u^6\,du$$

$$= \frac{1}{3}\left[\frac{1}{5}u^5 - \frac{1}{7}u^7\right] + C$$

$$= \frac{1}{3}\left[\frac{1}{5}\sin^5(3x) - \frac{1}{7}\sin^7(3x)\right] + C$$

$\sec^2\theta = \tan^2\theta + 1$

Example: $\int \sec^4(x)\,dx$

$= \int \sec^2(x)\,\sec^2(x)\,dx$

$u = \tan(x)$

$du = \sec^2 x\,dx$

$= \int [\tan^2(x)+1]\,\sec^2(x)\,dx$

$= \int u^2 + 1\,du$

$= \frac{1}{3}u^3 + u + C$

$= \frac{1}{3}\tan^3(x) + \tan(x) + C$

Example: $\int \tan^4(x)\sec^4(x)\,dx = \int \tan^4(x)\,\sec^2(x)\,\sec^2(x)\,dx$

$= \int \tan^4(x)\left(\tan^2(x)+1\right)\sec^2(x)\,dx$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$= \int u^4(u^2+1)\,du = \int u^6 + u^4\,du$

$= \frac{1}{7}u^7 + \frac{1}{5}u^5 + C$

$= \frac{1}{7}\tan^7(x) + \frac{1}{5}\tan^5(x) + C$

Example: $\int \tan^5(x)\sec^3(x)\,dx = \int \tan^4(x) \sec^2(x) \tan(x)\sec(x)\,dx$

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$$= \int \left(\tan^2(x)\right)^2 \sec^2(x) \tan(x)\sec(x)\,dx$$

$$= \int \left[\sec^2(x) - 1\right]^2 \sec^2(x) \tan(x) \sec(x)\,dx$$

$$= \int (u^2-1)^2 u^2\,du$$

$$= \int (u^4 - 2u^2 + 1) u^2\,du$$

$$= \int u^6 - 2u^4 + u^2\,du = \frac{1}{7}u^7 - \frac{2u^5}{5} + \frac{1}{3}u^3 + C$$

$$= \frac{1}{7}\sec^7(x) - \frac{2}{5}\sec^5(x) + \frac{1}{3}\sec^3(x) + C$$

Example: $\int \sec(x)\, dx = \int \sec(x) \cdot \frac{\sec(x) + \tan(x)}{\tan(x) + \sec(x)}\, dx$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\tan(x) + \sec(x)}\, dx = \int \frac{1}{u}\, du$$

$$= \ln|u| + C$$

$u = \tan(x) + \sec(x)$

$du = (\sec^2(x) + \sec(x)\tan(x))\, dx$

$$\int \sec(x)\, dx = \ln|\tan(x) + \sec(x)| + C$$

Example: $\int \sec^3(x)\,dx = J$

D	I
$\sec(x)$	$\sec^2(x)$
$\sec(x)\tan(x)$	$\tan(x)$

$+$

$-\int$

$$J = \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$$

$$= \sec(x)\tan(x) - \int \sec(x)\left(\sec^2(x) - 1\right)\,dx$$

$$= \sec(x)\tan(x) - \int \sec^3(x) - \sec(x)\,dx$$

$$J = \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx$$

$$2J = \sec(x)\tan(x) + \int \sec(x)\,dx$$

$$\int \sec^3(x)\,dx = J = \frac{1}{2}\left[\sec(x)\tan(x) + \ln\left|\tan(x) + \sec(x)\right|\right] + C$$

Example: $\int \cot^2(x)\csc^2(x)\,dx$ $= \int -u^2\,du$

$= -\frac{1}{3}u^3 + C$

$= -\frac{1}{3}\cot^3(x) + C$

$u = \cot(x)$

$du = -\csc^2(x)\,dx$

$-du = \csc^2(x)\,dx$