Section 7.3: Trigonometric Substitution

Comparison of two integrals.

$$\int x\sqrt{1-x^2}\,dx = \int \frac{-1}{2}u^{1/2}du = \frac{-1}{2} * \frac{2}{3}u^{3/2} + C = \frac{-1}{3}(1-x^2)^{3/2} + C$$

$u = 1 - x^2 \qquad \frac{-1}{2}du = x\,dx$

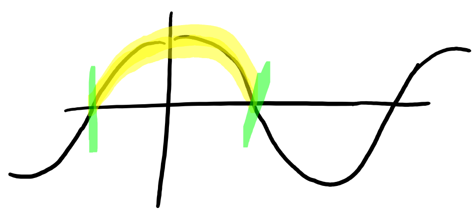
$$\int \sqrt{1-x^2}\,dx$$

Examine: $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$

need
$x^2 = \sin^2\theta$

$x = \sin\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



Some useful integrals.

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$

$$\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$$\int \csc^3 x \, dx = \frac{-1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$$

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ \frac{-\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Compute these integrals.

Example: $\int \sqrt{16 - x^2}\, dx$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$16\cos^2\theta = 16 - 16\sin^2\theta$

need

$x^2 = 16\sin^2\theta$

$x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$$\int \sqrt{16 - x^2}\, dx$$

$$= \int \sqrt{16 - 16\sin^2\theta} \cdot 4\cos\theta\, d\theta$$

$$= \int \sqrt{16\cos^2\theta} \cdot 4\cos\theta\, d\theta$$

$$= \int 4\cos\theta \cdot 4\cos\theta\, d\theta$$

$$= \int 16\cos^2\theta\, d\theta = \int 16 \cdot \frac{1}{2}\left[1 + \cos 2\theta\right] d\theta$$

$$= 8\int 1 + \cos 2\theta\, d\theta$$

$$= 8\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

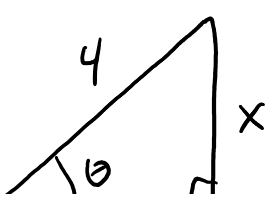
$$= 8\left[\theta + \cancel{\frac{1}{2}} \cdot \cancel{2}\sin\theta\cos\theta\right] + C$$

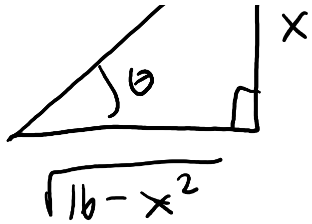
$$= \boxed{8\left[\arcsin\left(\frac{x}{4}\right) + \frac{x}{4} \cdot \frac{\sqrt{16 - x^2}}{4}\right] + C}$$

$x = 4\sin\theta$

$\frac{x}{4} = \sin\theta$

$\theta = \arcsin\left(\frac{x}{4}\right)$





$$\cos\theta = \frac{\sqrt{16-x^2}}{4}$$

Example: $\int \frac{1}{(x^2-9)^{3/2}}\,dx$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$9\tan^2\theta = 9\sec^2\theta - 9$ $\longleftrightarrow$ $\tan^2\theta = \sec^2\theta - 1$

need
$x^2 = 9\sec^2\theta$

let
$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$$= \int \frac{1}{(9\tan^2\theta)^{3/2}} \cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{3\sec\theta\tan\theta}{3^3\tan^3\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{9}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta$$

$$= \frac{1}{9}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \frac{1}{9}\int \frac{1}{u^2}\,du = \frac{1}{9}\left(\frac{-1}{u}\right) + C$$

$$= \frac{-1}{9\sin\theta} + C = \frac{-1}{9\frac{\sqrt{x^2-9}}{x}} + C$$

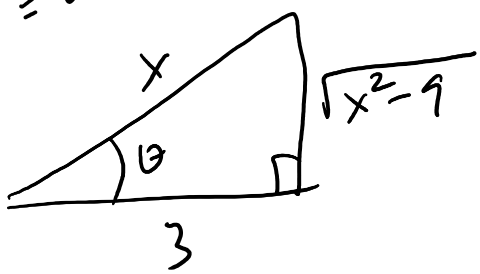
$X = 3\sec\theta$

$\frac{X}{3} = \sec\theta$

$\frac{3}{x} = \cos\theta$

$$= \boxed{\frac{-x}{9\sqrt{x^2-9}} + C}$$

$\frac{3}{x} = \cos\theta$



$$\sin\theta = \frac{\sqrt{x^2-9}}{x}$$

$$= \boxed{\frac{\quad}{9\sqrt{x^2-9}} + C}$$

Example: $\displaystyle\int \frac{1}{x^2\sqrt{16-9x^2}}\,dx$

$\cos^2\theta = 1-\sin^2\theta$

$\sec^2\theta = 1+\tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$16\cos^2\theta = 16 - 16\sin^2\theta$

need

$9x^2 = 16\sin^2\theta$

let

$3x = 4\sin\theta$

$x = \frac{4}{3}\sin\theta$

$dx = \frac{4}{3}\cos\theta\, d\theta$

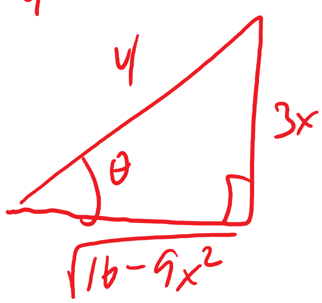
$$\int \frac{1}{\left(\frac{4}{3}\sin\theta\right)^2\sqrt{16\cos^2\theta}} \cdot \frac{4}{3}\cos\theta\, d\theta$$

$$= \int \frac{\frac{4}{3}\cos\theta}{\left(\frac{4}{3}\right)^2\sin^2\theta \cdot 4\cos\theta}\, d\theta$$

$$= \frac{1}{\frac{16}{3}}\int \frac{1}{\sin^2\theta}\, d\theta = \frac{3}{16}\int \csc^2\theta\, d\theta$$

$$= \frac{-3}{16}\cot\theta + C = \frac{-3}{16}\cdot\frac{\sqrt{16-9x^2}}{3x} + C$$

$\frac{3x}{4} = \sin\theta$



Example: $\int \frac{1}{x^2 + A^2}\, dx$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$A^2\sec^2\theta = A^2 + A^2\tan^2\theta$

want

$x^2 = A^2\tan^2\theta$

$x = A\tan\theta$

$dx = A\sec^2\theta\, d\theta$

$\frac{x}{A} = \tan\theta$

$$\int \frac{1}{A^2\sec^2\theta} \cdot A\sec^2\theta\, d\theta = \int \frac{1}{A}\, d\theta$$

$$= \frac{1}{A}\theta + C = \frac{1}{A}\arctan\left(\frac{x}{A}\right) + C$$

$$\boxed{\int \frac{1}{x^2+A^2}\, dx = \frac{1}{A}\arctan\left(\frac{x}{A}\right) + C}$$

Page 7: Complete the square.

Review of completing the square:

$x^2 + 8x =$ $x^2 + 8x + \underline{4^2} - \underline{4^2} = (x^2 + 8x + 4^2) - 4^2$

need a 1 in front of x^2

take $\frac{1}{2}$ and square.

$= (x+4)^2 - 16$

$4x^2 + 24x + 11 =$ $4[x^2 + 6x] + 11$

$= 4[x^2 + 6x + \underline{3^2} - \underline{3^2}] + 11$

$= 4(x+3)^2 - 4 \cdot 3^2 + 11$

$= 4(x+3)^2 - 36 + 11$

$= 4(x+3)^2 - 25$

Compute these integrals.

Example: $\displaystyle\int \frac{x}{\sqrt{4x-x^2}}\,dx$

$\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$$-x^2 + 4x = -1\left[x^2 - 4x\right] = -1\left[x^2 - 4x + 2^2 - 2^2\right]$$

$$= -1\left[(x-2)^2 - 4\right]$$

$$= 4 - (x-2)^2$$

$$\int \frac{x}{\sqrt{4-(x-2)^2}}\,dx$$

$$4\cos^2\theta = 4 - 4\sin^2\theta$$

need $(x-2)^2 = 4\sin^2\theta$

let $x - 2 = 2\sin\theta$

$x = 2 + 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

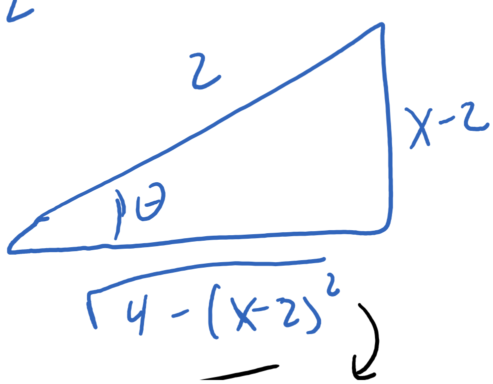
$$\int \frac{2+2\sin\theta}{\sqrt{4\cos^2\theta}}\, 2\cos\theta\, d\theta$$

$$= \int \frac{2+2\sin\theta}{2\cos\theta} \cdot 2\cos\theta\, d\theta$$

$$= \int 2 + 2\sin\theta\, d\theta$$

$$= 2\theta - 2\cos\theta + C$$

$\dfrac{x-2}{2} = \sin\theta$



$$= 2\arcsin\left(\frac{x-2}{2}\right) - 2\,\frac{\sqrt{4x-x^2}}{2} + C$$

$\sqrt{4-(x-2)^2}$

$\sqrt{4x-x^2}$