Math 152 Week in Review: Section 10.1, 10.2

1. Determine if the point is on the parametric curve $x(t) = t^2 - t + 1$, $y(t) = t - 2$

   (a) $(57, 6)$

   $6 = t - 2$ $\quad x(8) = 64 - 8 + 1 = 65 - 8 = 57$ ✓

   $8 = t$

   yes.

   (b) $(40, 5)$

   $t - 2 = 5$ $\quad x(7) = 7^2 - 7 + 1 = 49 - 7 + 1 \quad 50 - 7$

   $t = 7$ $\quad = 43$ ✗

   no

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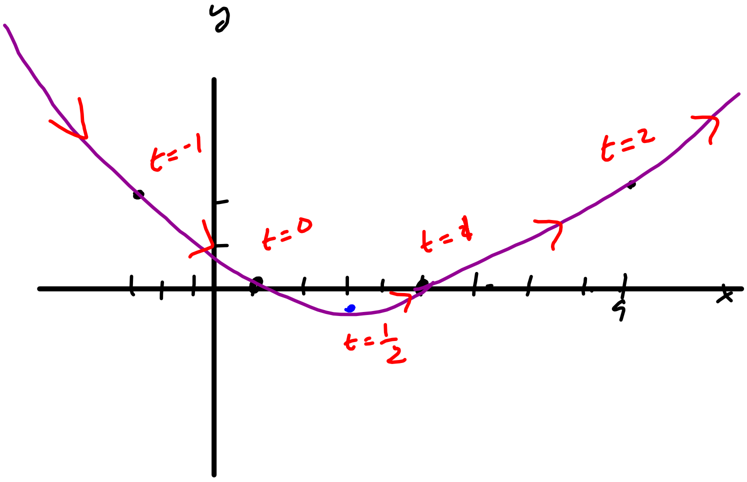
2. For each of the following parametric equations sketch the curve and indicate with an arrow the direction in which the curve increases as t increases. Then eliminate the parameter to find a a Cartesian equation of the curve.

(a) $x(t) = 1 + 4t, \qquad y(t) = t^2 - t$

t	x	y
-2	-7	6
-1	-3	2
0	1	0
1	5	0
2	9	2
3	13	6
$\frac{1}{2}$	3	$-\frac{1}{4}$



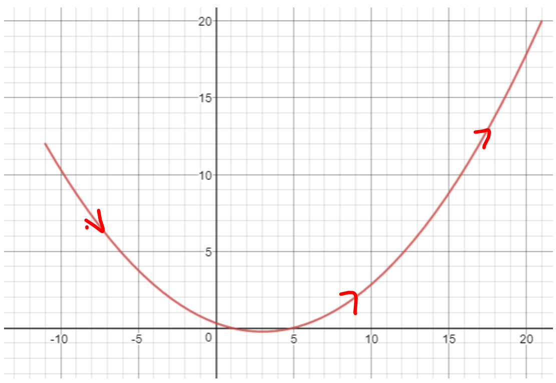
$x = 1 + 4t$

$x - 1 = 4t$

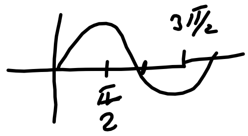
$\frac{x-1}{4} = t$

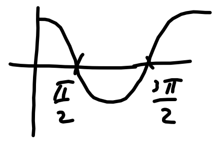
$y = t^2 - t$

$y = \left(\frac{x-1}{4}\right)^2 - \left(\frac{x-1}{4}\right)$

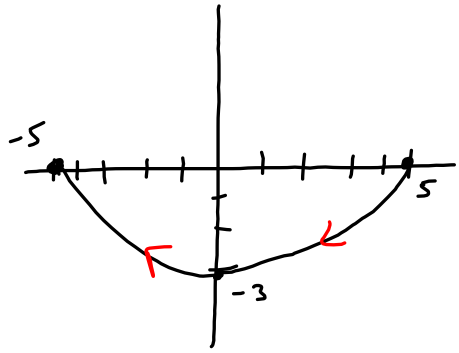


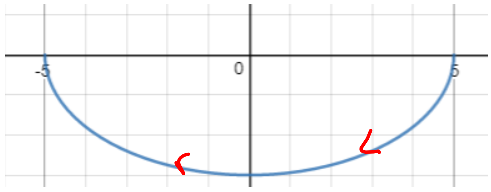
(b) $x = 5\sin\theta, \quad y = 3\cos\theta, \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$





θ	x	y
$\frac{\pi}{2}$	5	0
π	0	-3
$\frac{3\pi}{2}$	-5	0





$\frac{x}{5} = \sin\theta \qquad \frac{y}{3} = \cos\theta$

$\sin^2\theta + \cos^2\theta = 1$

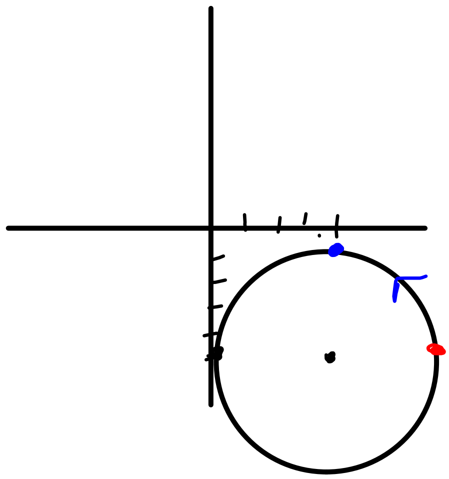
$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ ellipse

(c) $x(t) = 4 + 4\cos\theta, \quad y(t) = -5 + 4\sin\theta$

$\theta = 0 \quad x = 8 \quad y = -5$

$\theta = \frac{\pi}{2} \quad x = 4 \quad y = -1$

$$\frac{x-4}{4} = \cos\theta \qquad \frac{y+5}{4} = \sin\theta$$



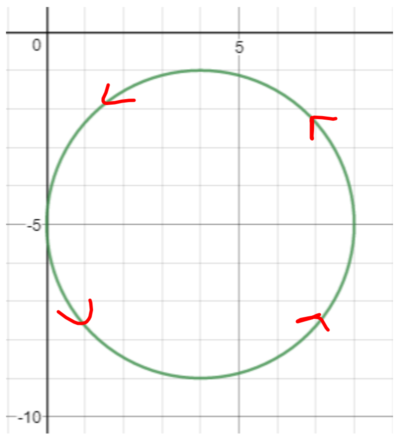
$$\cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{x-4}{4}\right)^2 + \left(\frac{y+5}{4}\right)^2 = 1$$

$$\frac{(x-4)^2}{16} + \frac{(y+5)^2}{16} = 1$$

$$(x-4)^2 + (y+5)^2 = 16$$

circle.



$$L = \int_a^b ds \qquad ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$x' = 2t \qquad y' = 3t^2$

3. Find the length of the arc of the curve $x = t^2$, $y = t^3$ that lies between the points $(1, 1)$ and $(4, 8)$.

$t=1 \qquad t=2$

$y=1 \qquad y=8$

$t^3=1 \qquad t^3=8$

$t=1 \qquad t=2$

$$L = \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt = \int_1^2 \sqrt{4t^2 + 9t^4}\, dt$$

$$= \int_1^2 \sqrt{t^2(4+9t^2)}\, dt = \int_1^2 |t| \sqrt{4+9t^2}\, dt$$

$$= \int_1^2 t\sqrt{4+9t^2}\, dt$$

$u = 4+9t^2$

$du = 18t\, dt$

$\frac{1}{18}du = t\, dt$

$$= \int_{t=1}^{t=2} \frac{1}{18}\sqrt{u}\, du = \int_{t=1}^{t=2} \frac{1}{18} u^{1/2}\, du = \frac{1}{18} u^{3/2} \cdot \frac{2}{3} \Big|_{t=1}^{t=2}$$

$$= \frac{1}{27}(4+9t^2)^{3/2} \Big|_1^2$$

$$= \boxed{\frac{1}{27}(40)^{3/2} - \frac{1}{27}(13)^{3/2}}$$

4. Find the length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ for $1 \le x \le 3$

$\frac{1}{2t} = \frac{1}{2}t^{-1}$

$-\frac{1}{2}t^{-2}$

$\frac{-1}{2t^2}$

$x = t \qquad y = \frac{t^3}{6} + \frac{1}{2t} \qquad 1 \le t \le 3$

$x' = 1 \qquad y' = \frac{3t^2}{6} - \frac{1}{2t^2}$

$$ds = \sqrt{(x')^2 + (y')^2} = \sqrt{1^2 + \left(\frac{t^2}{2} - \frac{1}{2t^2}\right)^2}$$

$$= \sqrt{1 + \frac{t^4}{4} - 2\frac{t^2}{2}\frac{1}{2t^2} + \frac{1}{4t^4}}$$

$$= \sqrt{1 + \frac{t^4}{4} - \frac{1}{2} + \frac{1}{4t^4}}$$

$$= \sqrt{\frac{t^4}{4} + \frac{1}{2} + \frac{1}{4t^2}}$$

$$= \sqrt{\left(\frac{t^2}{2} + \frac{1}{2t^2}\right)^2}$$

$$= \frac{t^2}{2} + \frac{1}{2t^2}$$

$$L = \int_1^3 ds = \int_1^3 \frac{t^2}{2} + \frac{1}{2t^2}\,dt = \int_1^3 \frac{t^2}{2} + \frac{1}{2}t^{-2}\,dt$$

$$= \frac{t^3}{6} + \frac{1}{2}\frac{t^{-1}}{(-1)}\Big|_1^2 = \frac{t^3}{6} - \frac{1}{2t}\Big|_1^3$$

$$= \frac{27}{6} - \frac{1}{6} - \left(\frac{1}{6} - \frac{1}{2}\right)$$

$$= \frac{25}{6} + \frac{1}{2}$$

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5. Find the length of the curve $x = e^t - t$, $y = 4e^{t/2}$, $0 \le t \le 2$

$x' = e^t - 1$ $\qquad$ $y' = 4 \cdot \frac{1}{2} e^{t/2} = 2e^{t/2}$

$$ds = \sqrt{(e^t-1)^2 + (2e^{t/2})^2}$$

$$= \sqrt{(e^t-1)(e^t-1) + 4e^t}$$

$$= \sqrt{e^{2t} - 2e^t + 1 + 4e^t} = \sqrt{e^{2t} + 2e^t + 1}$$

$$= \sqrt{(e^t+1)^2} = e^t + 1$$

$$L = \int_0^2 ds = \int_0^2 e^t + 1 \, dt = e^t + t \Big|_0^2$$

$$= e^2 + 2 - (e^0 + 0) = e^2 + 2 - 1$$

$$= e^2 + 1$$

6. Find the length of the curve $x = e^t \sin(t)$, $y = e^t \cos(t)$, $0 \leq t \leq \pi$

$$x' = e^t \sin(t) + e^t \cos(t)$$
$$= e^t(\sin t + \cos t)$$

$$y' = e^t \cos(t) - e^t \sin(t)$$
$$= e^t(\cos(t) - \sin t)$$

$$ds = \sqrt{(x')^2 + (y')^2}$$
$$= \sqrt{[e^t(\sin t + \cos t)]^2 + [e^t(\cos t - \sin t)]^2}$$
$$= \sqrt{e^{2t}(\sin t + \cos t)^2 + e^{2t}(\cos t - \sin t)^2}$$
$$= \sqrt{e^{2t}[(\sin t + \cos t)^2 + (\cos t - \sin t)^2]}$$
$$= e^t\sqrt{\underline{\underline{\sin^2 t}} + \cancel{2\sin t \cos t} + \underline{\underline{\cos^2 t}} + \underline{\underline{\cos^2 t}} - \cancel{2\cos t \sin t} + \underline{\underline{\sin^2 t}}}$$
$$= e^t\sqrt{1 + 1} = e^t\sqrt{2}$$

$$L = \int_0^\pi ds = \int_0^\pi e^t\sqrt{2}\, dt = e^t\sqrt{2}\Big|_0^\pi$$
$$= e^\pi\sqrt{2} - e^0\sqrt{2}$$
$$= e^\pi\sqrt{2} - \sqrt{2}$$

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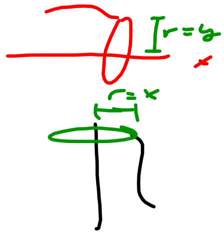
$$SA = \int_a^b 2\pi r \, ds$$

about x-axis

$$\int_a^b 2\pi y \, ds$$

about y-axis

$$\int_a^b 2\pi x \, ds$$



7. Find the area of the surface obtained by rotating the curve about the x–axis. $r = y$

$x = \frac{t^3}{3}$, $y = t^2$, $0 \le t \le 1$ $\qquad x' = \frac{3t^2}{3} = t^2 \qquad y' = 2t$

$$ds = \sqrt{(x')^2 + (y')^2} = \sqrt{t^4 + 4t^2} = \sqrt{t^2(t^2+4)} = t\sqrt{t^2+4}$$

$$SA = \int_0^1 2\pi r \, ds = \int_0^1 2\pi y \cdot t\sqrt{t^2+4} \, dt$$

$$= \int_0^1 2\pi \, t^2 \cdot t \sqrt{t^2+4} \, dt$$

$u = t^2+4$

$du = 2t \, dt$

$\frac{1}{2} du = t \, dt$

$u - 4 = t^2$

$$= \int_4^5 2\pi (u-4) \tfrac{1}{2} \sqrt{u} \, du$$

$$= \pi \int_4^5 u^{3/2} - 4u^{1/2} \, du = \pi \left[\tfrac{2}{5} u^{5/2} - 4u^{3/2} \cdot \tfrac{2}{3} \right]_4^5$$

$$\boxed{= \pi \left[\tfrac{2}{5}(5)^{5/2} - \tfrac{8}{3}(5)^{3/2} - \left(\tfrac{2}{5}(4)^{5/2} - \tfrac{8}{3}(4)^{3/2} \right) \right]}$$

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8. Find the area of the surface obtained by rotating the curve about the $y-$axis.

$x = 5\sin t,\ y = 5\cos t,\ 0 \le t \le \pi$

$$SA = \int_a^b 2\pi x \, ds$$

$x' = 5\cos t \qquad y' = -5\sin t$

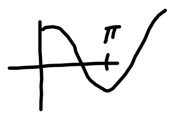
$$ds = \sqrt{(x')^2 + (y')^2} = \sqrt{25\cos^2 t + 25\sin^2 t}$$

$$= \sqrt{25(\cos^2 t + \sin^2 t)} = \sqrt{25} = 5$$

$$SA = \int_0^{\pi} 2\pi \, 5\sin t \cdot 5 \, dt = 50\pi \int_0^{\pi} \sin t \, dt$$

$$= 50\pi \left[-\cos t \right]_0^{\pi} = 50\pi \left[-\cos\pi - -\cos(0) \right]$$

$$= 50\pi \left[1 + 1 \right] = 100\pi$$



9. Find the area of the surface obtained by rotating the curve about the $y-$axis.

$$SA = \int_a^b 2\pi x \, ds$$

$x = 3t^2$, $y = 2t^3$, $0 \le t \le 5$

$x' = 6t \qquad y' = 6t^2$

$$ds = \sqrt{(x')^2 + (y')^2} = \sqrt{36t^2 + 36t^4}$$

$$= \sqrt{36t^2(1+t^2)} = 6t\sqrt{1+t^2}$$

$$SA = \int_0^5 2\pi \, 3t^2 \cdot 6t\sqrt{1+t^2} \, dt$$

$u = 1+t^2$

$du = 2t \, dt$

$\frac{1}{2} du = t \, dt$

$u - 1 = t^2$

$$= \int_1^{26} 2\pi \cdot 3(u-1) \cdot 6\sqrt{u} \cdot \frac{1}{2} du$$

$$= \int_1^{26} 18\pi \left(u^{3/2} - u^{1/2} \right) du$$

$$= 18\pi \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_1^{26}$$

$$= 18\pi \left[\frac{2}{5}(26)^{5/2} - \frac{2}{3}(26)^{3/2} - \left(\frac{2}{5} - \frac{2}{3} \right) \right]$$

10. Setup the integral that would find the area of the surface obtained by rotating the curve about the $x-$axis.

$x = 2t - t^2$, $y = 3 + t^2$, $0 \leq t \leq 2$

$$\int 2\pi y \, ds$$

$$\int_0^2 2\pi(3+t^2)\sqrt{(2-2t)^2 + (2t)^2}\, dt$$

11. Setup the integral that would find the area of the surface obtained by rotating the curve about the $y-$axis.

$x = 2t - t^2$, $y = 3 + t^2$, $0 \leq t \leq 2$

$$\int 2\pi x \, ds$$

$$\int_0^2 2\pi(2t-t^2)\sqrt{(2-2t)^2 + (2t)^2}\, dt$$