Math 152 Exam 3 Review

Solutions and questions can be found at the link:
https://www.math.tamu.edu/~kahlig/152WIR.html

The following is a collection of questions to review the topics for the second exam. This is not intended to represent an actual exam nor does it have every type of problem seen int he homework.
These questions cover sections 11.4, 11.5, 11.6, 11.8, 11.9, 11.10, 11.11

Important Maclaurin series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \qquad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots \qquad R = \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad R = \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \qquad R = \infty$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \qquad -1 < x \leq 1$$

1. Determine if the series converges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{n+1} n^4}{9^{n+3}}$

$$a_n = \frac{(-1)^{n+1}\, 5^{n+1}\, n^4}{9^{n+3}}$$

$$a_{n+1} = \frac{(-1)^{n+2}\, 5^{n+2}\, (n+1)^4}{9^{n+4}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+2}}\, 5^{n+2}\, (n+1)^4}{9^{n+4}} \cdot \frac{9^{n+3}}{\cancel{(-1)^{n+1}}\, 5^{n+1}\, n^4} \right|$$

$$\lim_{n\to\infty} \frac{5\,(n+1)^4}{9\, n^4} = \lim_{n\to\infty} \frac{5}{9} \left(\boxed{\frac{n+1}{n}} \right)^4 = \frac{5}{9}$$

The series is Abs. Conv. (ie convergent.)

(b) $\sum_{n=1}^{\infty} \frac{7n^3 + \cos(2n)}{n^4+1}$

$$-1 \le \cos(2n) \le 1$$

$$0 < 7n^3 - 1 \le 7n^3 + \cos(2n) \le 7n^3 + 1$$

$$\frac{7n^3-1}{n^4+1} \le \frac{7n^3+\cos(2n)}{n^4+1} \le \frac{7n^3+1}{n^4+1}$$

$$\sum \frac{7n^3-1}{n^4+1} \qquad \sum \frac{7n^3+1}{n^4+1}$$

Look like $\sum \frac{7n^3}{n^4} = \sum \frac{7}{n}$ p-series $p=1$ div.

use LCT with $\sum \frac{7}{n}$

$$\lim_{n\to\infty} \frac{\frac{7n^3-1}{n^4+1}}{\frac{7}{n}} = \lim_{n\to\infty} \frac{7n^4-n}{7n^4+7} = 1$$

by LCT $\sum \frac{7n^3-1}{n^4+1}$ div.

by the comparison test we get

$$\sum_{n=1} \frac{7n^3+\cos(2n)}{n^4+1}$$ will div.

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test for div.

$$\lim_{n\to\infty} \frac{7n^3+\cos(2n)}{n^4+1} = \lim_{n\to\infty} \frac{7n^3}{n^4+1} + \frac{\cos(2n)}{n^4+1} = 0$$

Test for div does not help.

(c) $\sum_{n=1}^{\infty} \frac{4^n}{3^{2n}-7} = \sum \frac{4^n}{9^n-7}$

Comparison is the wrong direction

$$9^n > 9^n - 7$$

$$\frac{1}{9^n} < \frac{1}{9^n-7}$$

$$\frac{4^n}{9^n} < \frac{4^n}{9^n-7}$$

use LCT $\sum \left(\frac{4}{9}\right)^n = \sum \frac{4^n}{9^n}$ geometric $r = \frac{4}{9}$ conv.

$$\lim_{n\to\infty} \frac{\frac{4^n}{9^n-7}}{\frac{4^n}{9^n}} = \lim_{n\to\infty} \frac{4^n}{9^n-7}\cdot\frac{9^n}{4^n} = \lim_{n\to\infty}\frac{9^n}{9^n-7}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{9^n \ln(9)}{9^n \ln(9)} = 1$$

by LCT $\sum \frac{4^n}{9^n-7}$ Converges.

2. Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-2)^n$ has a radius of convergence of 7. What can be concluded about the convergence/divergence of the following pair of series?

$$\text{(I)} \sum_{n=0}^{\infty} (-1)^n c_n 4^n \qquad \text{(II)} \sum_{n=0}^{\infty} c_n 9^n$$

Converges for

$-5 < x < 9$

↳ $\sum c_n(-4)^n$

I

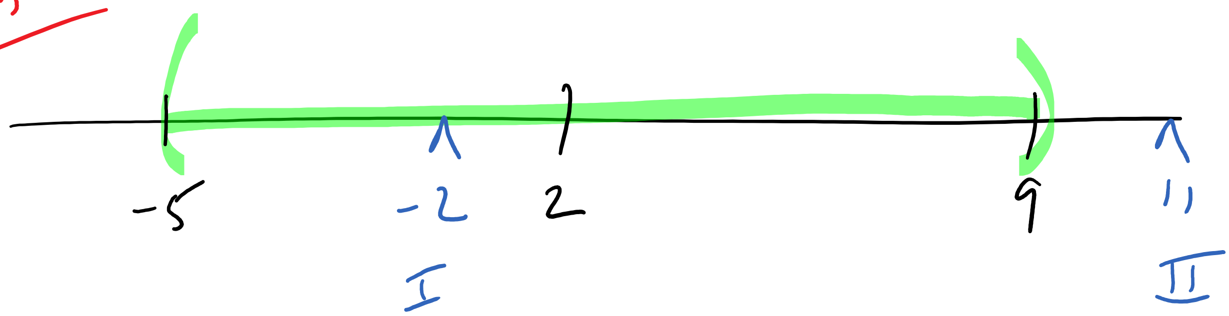
$x-2=-4$
$x=-2$

Converges

II

$x-2=9$
$x=11$

diverges



3. Consider the 5th partial sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n7^n}$ as an approximation. Use the alternating series rule to obtain an upper bound on the absolute value of the error.

$\sum_{n=1} (-1)^n b_n$

S_5 error is R_5

$b_n = \frac{1}{n7^n}$

max value of error $|R_n| \le b_{n+1}$

$|R_5| \le b_6 = \frac{1}{6 \cdot 7^6}$

4. Assume that the series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ will converge by the alternating series test.

Which of these is an approximation to the sum of the series so that the maximum error will be less than 0.001 and contains the fewest number of terms?

$$|R_n| \le b_{n+1} < .001$$

(a) $b_1 + b_2 + b_3 + b_4 + b_5$

$\boxed{\text{(b) } b_1 - b_2 + b_3 - b_4 + b_5}$

~~(c)~~ $-b_1 + b_2 - b_3 + b_4 - b_5$

~~(d)~~ $b_1 + b_2 + b_3 + b_4 + b_5 + b_6$

~~(e)~~ $b_1 - b_2 + b_3 - b_4 + b_5 - b_6$

~~(f)~~ $-b_1 + b_2 - b_3 + b_4 - b_5 + b_6$

n	b_n
1	0.5
2	0.0625
3	0.01388889
4	0.00390625
5	0.00125
6	0.00043403
7	0.00015944
8	0.00006104

$b_{n+1} = b_6$

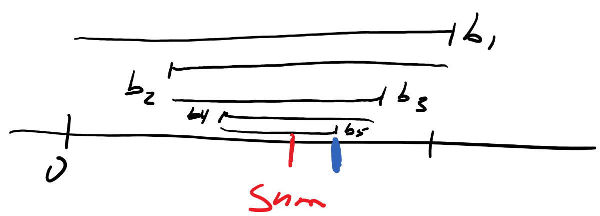
$n+1 = 6$

$\underline{n = 5}$

Is the approximation more or less than the actual sum?

(a) more

(b) less



5. Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^2} = A$

by the definition of Abs. conv. The series A is Absolutely Convergent.

by the definition

new series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n e^{1/n}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$$

Try LCT with $\sum \frac{1}{n^2}$ conv. p-series $p=2$

$$\lim_{n\to\infty} \frac{\frac{e^{1/n}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{e^{1/n}}{n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} e^{\frac{1}{n}} = e^0 = 1$$

by LCT $\sum \frac{e^{1/n}}{n^2}$ will converge.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/5}}$

new series

$$\sum_{n=1} \left| \frac{(-1)^n}{n^{3/5}} \right| = \sum_{n=1} \frac{1}{n^{3/5}}$$

p-series $p = 3/5$ div.

This is not Absolutely convergent.

use AST $b_n = \frac{1}{n^{3/5}}$

$b_n \to 0$ as $n \to \infty$

b_n are decreasing

by AST this series converges

The series is conditionally convergent

(c) $\sum_{n=2}^{\infty} \frac{7\sin(n^2+1)}{n^4-2n+5}$

new series

$$\sum_{n=2} \left| \frac{7\sin(n^2+1)}{n^4-2n+5} \right| = \sum_{n=2} \frac{|7\sin(n^2+1)|}{n^4-2n+5} = J$$

$$0 \le |7\sin(n^2+1)| \le 7$$

$$0 \le \frac{|7\sin(n^2+1)|}{n^4-2n+5} \le \frac{7}{n^4-2n+5}$$

now look at $\sum \frac{7}{n^4-2n+5}$ Looks like $\sum \frac{7}{n^4}$ p-series $p=4$ conv.

use LCT to show J will conv.

$$\lim_{n\to\infty} \frac{\frac{7}{n^4-2n+5}}{\frac{7}{n^4}} = \lim_{n\to\infty} \frac{7}{n^4-2n+5} \cdot \frac{n^4}{7} = \lim_{n\to\infty} \frac{n^4}{n^4-2n+5} = 1$$

by LCT $\sum \frac{7}{n^4-2n+5}$ conv.

by Comparison the series J will conv.

Thus the original series is Abs, conv.

6. Find the interval and radius of convergence of the series $\sum_{n=1}^{\infty} \frac{x^{2n}}{n^2\, 25^n}$.

Centered at $x=0$

$(x-a)^n \to$ center @ a

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(n+1)^2\, 25^{n+1}} \cdot \frac{n^2\, 25^n}{x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(n+1)^2\, 25^{n+1}} \cdot \frac{n^2\, 25^n}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2\, n^2}{(n+1)^2\, 25} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{25} \cdot \left(\frac{n}{n+1}\right)^2 \right| = \left| \frac{x^2}{25} \right| < 1$$

$$\left| \frac{x^2}{25} \right| < 1 \qquad \sqrt{x^2} = |x|$$

$$x^2 = |x^2| < 25$$

$$|x| < 5 \qquad R = 5$$

$$-5 < x < 5$$

now Test The endpoints for Interval of conv.

$$\sum_{n=1} \frac{x^{2n}}{n^2 25^n}$$

$x = 5 \,|\quad \sum \frac{5^{2n}}{\dots} = \sum \frac{25^n}{\dots}$

$\boxed{x=5}$ $\sum_{n=1} \frac{5^{2n}}{n^2 25^n} = \sum_{n=1} \frac{25^n}{n^2 25^n}$

$n=1$

$= \sum_{n=1} \frac{1}{n^2}$ p-series $p=2$ conv.

$\boxed{x=-5}$ $\sum \frac{(-5)^{2n}}{n^2 25^n} = \sum \frac{25^n}{n^2 25^n} = \sum \frac{1}{n^2}$ p-series $p=2$ conv.

Answer $R=5$ $I: [-5, 5]$

7. Find the radius of convergence and the interval of convergence of the power series.
$\sum_{n=1}^{\infty} \frac{(-1)^n(x+4)^n}{n7^n}$

centered at $x=-4$ $(a=-4)$

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}(x+4)^{n+1}}{(n+1)7^{n+1}}\cdot\frac{n7^n}{\cancel{(-1)^n}(x+4)^n}\right| = \lim_{n\to\infty}\left|\frac{(x+4)}{7}\frac{n}{n+1}\right|$$

$$=\left|\frac{x+4}{7}\right|<1$$

$$\left|\frac{x+4}{7}\right|<1$$

$|x-a|<R$

$|x--4| = |x+4| \leq R$

$$|x+4|<7 \longrightarrow R=7$$

$$-7<x+4<7$$

$$-11<x<3$$

centered at $a=-4$

$3--4 = 3+4 = 7 = R$

Test endpoints

$x=3$ $\sum_{n=1} \frac{(-1)^n(3+4)^n}{n7^n} = \sum_{n=1}\frac{(-1)^n 7^n}{n7^n} = \sum_{n=1}\frac{(-1)^n}{n}$

Conv. by AST $b_n=\frac{1}{n}$

$\boxed{x = -11}$ $\displaystyle\sum_{n=1} \frac{(-1)^n(-11+4)^n}{n\,7^n} = \sum_{n=1} \frac{(-1)^n(-7)^n}{n\,7^n}$

$$= \sum_{n=1} \frac{(-1)^n(-1)^n 7^n}{n\,7^n} = \sum_{n=1} \frac{1}{n}$$

div. by p-series $p=1$

$R = 7$ $\quad I : (-11, 3]$

8. Find the interval and radius of convergence of the series $\sum_{n=1}^{\infty} \frac{n!(2x-3)^n}{5^n}$.

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(2x-3)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n!\,(2x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\,(2x-3)}{5} \right| = \begin{cases} \infty & \text{if } x \neq \frac{3}{2} \\ 0 & \text{if } x = \frac{3}{2} \end{cases}$$

$R = 0 \qquad I = \left\{ \frac{3}{2} \right\}$

9. Find the sum of these series

$$\sin(x) = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!}$$

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{(2n+1)!} = \sin(5)$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{3^{2n}(2n)!} = \sum_{n=0} \frac{(-1)^n 5^{2n} \cdot 5^1}{3^{2n}(2n)!} = 5 \sum_{n=0} \frac{(-1)^n \left(\frac{5}{3}\right)^{2n}}{(2n)!}$

$$= 5\cos\left(\frac{5}{3}\right)$$

$$e^x = \sum_{n=0} \frac{x^n}{n!}$$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n}{n!} = e^{-2} - \underbrace{\frac{(-2)^0}{0!}}_{n=0} - \underbrace{\frac{(-2)^1}{1!}}_{n=1} = e^{-2} - 1 - -2$

$$= e^{-2} - 1 + 2$$

$$= e^{-2} + 1$$

missing
$n=0$ term
$n=1$ term

10. Find the Maclaurin series for the function $f(x) = \dfrac{x^2}{(1-3x)^2}$. $= \frac{1}{3} \cdot x^2 \cdot g'$

let $g = \dfrac{1}{1-3x} = (1-3x)^{-1} = \sum_{n=0} (3x)^n = \sum_{n=0} 3^n x^n$

$|3x|<1$
$|x|< \frac{1}{3} = R$

$g' = -(1-3x)^{-2} \cdot (-3)$
$= \dfrac{3}{(1-3x)^2} = \sum_{n=1} 3^n n x^{n-1}$

$f = \dfrac{x^2}{3} g' = \dfrac{x^2}{3} \sum_{n=1} 3^n n x^{n-1}$

$$\boxed{f = \sum_{n=1} 3^{n-1} n x^{n+1}}$$

11. Use a MacLaurin series for $f(x) = x^3 \arctan(5x^2)$ to answer the following.

$\arctan(5x^2)$

$$= \sum_{n=0} \frac{(-1)^n (5x^2)^{2n+1}}{2n+1}$$

$$\arctan(5x^2) = \sum_{n=0} \frac{(-1)^n 5^{2n+1} x^{4n+2}}{2n+1}$$

$$x^3 \arctan(5x^2) = \sum_{n=0} \frac{(-1)^n 5^{2n+1} x^{4n+5}}{2n+1}$$

$5x^5 - , \ldots$

(a) $f'(x) =$

$$= \sum_{n=0} \frac{(-1)^n 5^{2n+1} (4n+5) x^{4n+4}}{2n+1}$$

(b) $\int f(x)dx =$ $C + \sum_{n=0} \frac{(-1)^n 5^{2n+1} x^{4n+6}}{(2n+1)(4n+6)}$

$\sum c_n (x-a)^n$

12. Find the 23th derivative of $f(x)$ at $x = 5$, i.e. $f^{(23)}(5)$, for $f(x) = \sum_{n=0}^{\infty} \boxed{\frac{(-1)^n}{3^n(n+2)}} (x-5)^n$

centered at
$x = 5$
$(a = 5)$

$$\underbrace{c_n = \frac{(-1)^n}{3^n(n+2)} = \frac{f^{(n)}(5)}{n!}}$$

$$f^{(n)}(5) = \frac{(-1)^n \, n!}{3^n (n+2)}$$

$$f^{(23)}(5) = \frac{-(23)!}{3^{23}(25)}$$

13. Find the Taylor series for $f(x) = \frac{1}{x^2}$ about $a = 5$. Express your answer in summation notation.

$n=0$

$f(x) = x^{-2} = \frac{1}{x^2}$

$f'(x) = -2x^{-3} = \frac{-2}{x^3}$

$f''(x) = 2\cdot 3x^{-4} = \frac{2\cdot 3}{x^4}$

$f'''(x) = -2\cdot 3\cdot 4x^{-5} = \frac{-2\cdot 3\cdot 4}{x^5}$

$f^{(4)}(x) = 2\cdot 3\cdot 4\cdot 5x^{-6} = \frac{2\cdot 3\cdot 4\cdot 5}{x^6}$

$f^{(n)}(x) = \frac{(-1)^n (n+1)!}{x^{n+2}}$

works for $n \geq 0$

find C_n

$C_n = \frac{1}{n!} f^{(n)}(5)$

$C_n = \frac{1}{n!} \cdot \frac{(-1)^n (n+1)!}{5^{n+2}}$

$C_n = \frac{(-1)^n (n+1)}{5^{n+2}}$

$$f(x) = \sum_{n=0} \frac{(-1)^n (n+1)}{5^{n+2}} (x-5)^n$$

14. Find the Taylor series for $f(x) = (x+1)e^x$ about $a = 2$. Express your answer in summation notation.

$$f(x) = (x+1)e^x$$

$$f'(x) = 1e^x + (x+1)e^x = [1+(x+1)]e^x$$

$$= (x+2)e^x$$

$$f''(x) = 1e^x + (x+2)\cdot e^x = [1+(x+2)]e^x$$

$$= (x+3)e^x$$

$$f'''(x) = (x+4)e^x$$

$$f^{(n)}(x) = (x+n+1)e^x$$

works for $n \geq 0$

$$c_n = \frac{f^{(n)}(2)}{n!}$$

$$c_n = \frac{(2+n+1)e^2}{n!}$$

$$c_n = \frac{(3+n)}{n!}e^2$$

$$\boxed{f(x) = \sum_{n=0} \frac{(3+n)e^2}{n!}(x-2)^n}$$

15. Find the 4th degree Taylor polynomial, $T_4(x)$, for $f(x) = x\ln(x)$ about $x = 6$.

$$T_4(x) = f(6) + \frac{f'(6)}{1!}(x-6) + \frac{f''(6)}{2!}(x-6)^2 + \frac{f'''(6)}{3!}(x-6)^3 + \frac{f^{(4)}(6)}{4!}(x-6)^4$$

$$f(x) = x\ln(x) \qquad f(6) = 6\ln(6)$$

$$f'(x) = 1\ln(x) + x \cdot \frac{1}{x} \qquad f'(6) = \ln(6) + 1$$

$$= \ln(x) + 1$$

$$f''(6) = \frac{1}{6}$$

$$f''(x) = \frac{1}{x} = x^{-1} \qquad f'''(6) = \frac{-1}{36}$$

$$f'''(x) = -x^{-2} = \frac{-1}{x^2}$$

$$f^{(4)}(x) = 2x^{-3} = \frac{2}{x^3} \qquad f^{(4)}(6) = \frac{2}{6^3}$$

$$T_4(x) = f(6) + \frac{f'(6)}{1!}(x-6) + \frac{f''(6)}{2!}(x-6)^2 + \frac{f'''(6)}{3!}(x-6)^3 + \frac{f^{(4)}(6)}{4!}(x-6)^4$$

$$= 6\ln(6) + [\ln(6)+1](x-6) + \frac{1}{2!}\cdot\frac{1}{6}(x-6)^2 + \frac{1}{3!}\cdot\left(\frac{-1}{36}\right)(x-6)^3 + \frac{1}{4!}\cdot\frac{2}{6^3}(x-6)^4$$