Solutions and questions can be found at the link:
https://www.math.tamu.edu/~kahlig/152WIR.html

1. Give the partial fraction decomposition of these fractions. Do not solve for the constants.

   (a) $\dfrac{x^4}{x^3+5x^2+6x}$

$$x^3+5x^2+6x+0 \overline{\big)\, x^4+0x^3+0x^2+0x+0}$$

$$\begin{array}{r} x-5 \\ -(x^4+5x^3+6x^2+0x) \\ \hline -5x^3-6x^2+0x+0 \\ -(-5x^3-25x^2-30x+0) \\ \hline 19x^2+30x \end{array}$$

$$= x-5+\frac{19x^2+30x}{x^3+5x^2+6x}$$

$$= x-5+\frac{\cancel{x}(19x+30)}{\cancel{x}(x^2+5x+6)}$$

$$= x-5+\frac{19x+30}{(x+2)(x+3)}$$

$$= \underbrace{x-5+\frac{A}{x+2}+\frac{B}{x+3}}$$

(b) $\dfrac{3x}{x(x-2)^3(x^2+9)^2} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2} + \dfrac{d}{(x-2)^3} + \dfrac{ex+f}{x^2+9} + \dfrac{gx+h}{(x^2+9)^2}$

multiplicity

(c) $\dfrac{x+1}{(x-2)(x^4-16)}$

$$x^4-16 = (x^2-4)(x^2+4)$$
$$= (x+2)(x-2)(x^2+4)$$

$$\frac{x+1}{(x+2)(x-2)^2(x^2+4)} = \frac{A}{x+2} + \frac{B}{(x-2)^1} + \frac{C}{(x-2)^2} + \frac{dx+e}{x^2+4}$$

2. $\int \frac{6x^2+x+4}{(x-2)(x^2+2)}\,dx$

$$\frac{6x^2+x+4}{\boxed{(x-2)(x^2+2)}} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2}$$

$$6x^2+x+4 = \left[\frac{A}{x-2} + \frac{Bx+C}{x^2+2}\right](x-2)(x^2+2)$$

$$6x^2+x+4 = A(x^2+2) + (Bx+C)(x-2)$$

$$6x^2+x+4 = Ax^2+2A + Bx^2-2Bx+Cx-2C$$

$$6x^2+x+4 = (A+B)x^2 + (-2B+C)x + 2A-2C$$

x^2 : $6 = A+B$

x : $1 = -2B+C$

Const : $4 = 2A-2C$

if $x=2$)

$6(4)+2+4 = A(4+2)$

$30 = 6A$

$5 = A$

$B=1$

$1 = -2+C$

$C=3$

2. $\int \frac{6x^2+x+4}{(x-2)(x^2+2)}\,dx = \int \frac{5}{x-2} + \frac{x+3}{x^2+2}\,dx$

$= \int \frac{5}{} \quad x \quad 3 \quad dx$

$$= \int \frac{5}{x-2} + \frac{x}{x^2+2} + \frac{3}{x^2+2}\, dx$$

$$= \underbrace{\int \frac{5}{x-2}\, dx}_{u = x-2} + \underbrace{\int \frac{x}{x^2+2}\, dx}_{u = x^2+2} + \int \frac{3}{x^2+2}\, dx$$

$$= 5 \ln|x-2| + \frac{1}{2} \ln|x^2+2| + \frac{3}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) + C$$

$$\boxed{\int \frac{1}{x^2+A^2}\, dx = \frac{1}{A} \arctan\left(\frac{x}{A}\right) + C}$$

3. $\int \frac{2x^3 - 5x^2 - 32}{x^2 - 4x}\,dx = J$

$$\begin{array}{r|l} & 2x + 3 \\ \hline x^2 - 4x + 0 & 2x^3 - 5x^2 + 0x - 32 \\ & -(2x^3 - 8x^2 + 0x) \\ \hline & 3x^2 + 0x - 32 \\ & -(3x^2 - 12x + 0) \\ \hline & 12x - 32 \end{array}$$

$$J = \int 2x + 3 + \frac{12x - 32}{x(x-4)}\,dx$$

$$\frac{12x - 32}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$$

$$12x - 32 = A(x-4) + Bx$$

if $x = 0$

$-32 = -4A$

$8 = A$

if $x = 4$

$48 - 32 = 4B$

$16 = 4B$

$B = 4$

$$J = \int 2x + 3 + \frac{8}{x} + \frac{4}{x-4}\,dx$$

$$= x^2 + 3x + 8\ln|x| + 4\ln|x-4| + C$$

4. $\int \frac{x^3-4x^2-11x-4}{(x+2)^2(x^2-x)}\,dx = J$

$(x+2)^2\, x(x-1)$

$$\frac{x^3-4x^2-11x-4}{(x+2)^2\,x(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x} + \frac{d}{x-1}$$

$$x^3-4x^2-11x-4 = A(x+2)x(x-1) + Bx(x-1) + C(x+2)^2(x-1) + d(x+2)^2x$$

$x=-2$ $\quad -8-16+22-4 = B(-2)(-3)$

$-6 = 6B \qquad \boxed{B=-1}$

$x=0$ $\quad -4 = C(+2)^2(-1)$

$-4 = -4C \longrightarrow \boxed{C=1}$

$x=1$ $\quad 1-4-11-4 = d(3)^2(1)$

$-18 = 9d \longrightarrow \boxed{d=-2}$

$x=-1$ $\quad -1-4+11-4 = A(1)(-1)(-2)+(-1)(-1)(-2)+1(1)^2(-2)+(-2)(1)(-1)$

$2 = 2A + (-2) + \cancel{(-2)} + \cancel{2}$

$4 = 2A$

$A=2$

$u = x+2$

$$J = \int \frac{2}{x+2} + \overbrace{\frac{-1}{(x+2)^2}}^{u = x+2} + \frac{1}{x} + \frac{-2}{x-1} \, dx$$

$$= 2\ln|x+2| + \frac{1}{x+2} + \ln|x| - 2\ln|x-1| + C$$

5. $\int \frac{5x^3+8x^2+25x+72}{x(x^2+9)(x^2+4)}\,dx = J$

$$\frac{5x^3+8x^2+25x+72}{x(x^2+9)(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+9} + \frac{dx+e}{x^2+4}$$

$$5x^3+8x^2+25x+72 = A(x^2+9)(x^2+4) + (Bx+C)\underbrace{x(x^2+4)}_{x^3+4x} + (dx+e)\underbrace{x(x^2+9)}_{x^3+9x}$$

$$= A(x^4+13x^2+36) + B(x^4+4x^2) + C(x^3+4x) + d(x^4+9x^2) + e(x^3+9x)$$

$$5x^3+8x^2+25x+72 = Ax^4+13Ax^2+36A + Bx^4+4Bx^2+Cx^3+4Cx + dx^4+9dx^2+ex^3+9ex$$

x^4 | $0 = A+B+d$

x^3 | $5 = C+e$

x^2 | $8 = 13A+4B+9d$

x | $25 = 4C+9e$

Const | $72 = 36A \longrightarrow A=2$

$C+e=5$

$C=5-e$

$25 = 4(5-e)+9e$

$25 = 20-4e+9e$

$5 = 5e$

$$\boxed{\begin{gathered}1=e\\ C=4\end{gathered}}$$

$0 = 2+B+d$

$-2-B = d$

$8 = 26+4B+9d$

$-18 = 4B+9(-2-B)$

$d = -2$

$-18 = 4B - 18 - 9B$

$0 = -5B \rightarrow B = 0$

$$J = \int \frac{2}{x} + \frac{4}{x^2+9} + \frac{-2x+1}{x^2+4}\,dx$$

$$= \int \frac{2}{x} + \frac{4}{x^2+9} + \frac{-2x}{\underbrace{x^2+4}_{u = x^2+4}} + \frac{1}{x^2+4}\,dx$$

$$= 2\ln|x| + \frac{4}{3}\arctan\left(\frac{x}{3}\right) - \ln|x^2+4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

$$\int \frac{-2x}{x^2+4}\,dx = \int -\frac{1}{u}\,du = -\ln|x^2+4| + C$$

$u = x^2+4 \quad du = 2x\,dx$

Determine if these integrals converge or diverge. if converge, then compute the value.

6. $\int_3^\infty \frac{x^2}{\sqrt[3]{x^3+1}}\,dx = \lim_{t\to\infty} \int_3^t \frac{x^2}{\sqrt[3]{x^3+1}}\,dx = \lim_{t\to\infty} \frac{1}{2}\sqrt[3]{(x^3+1)^2}\Big|_3^t$

$$\int \frac{x^2}{\sqrt[3]{x^3+1}}\,dx = \int \frac{1}{3}\frac{1}{\sqrt[3]{u}}\,du$$

$u = x^3+1$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$$= \frac{1}{3}\int u^{-1/3}\,du$$

$$= \frac{1}{3}\cdot\frac{3}{2}u^{2/3}$$

$$= \frac{1}{2}\sqrt[3]{(x^3+1)^2}$$

$$\lim_{t\to\infty}\left(\frac{1}{2}\sqrt[3]{(t^3+1)^2} - \frac{1}{2}\sqrt[3]{28^2}\right)$$

$$= \infty$$

The Integral diverges

(to ∞)

7. $\int_2^\infty 3xe^{-4x}\,dx = \lim_{t\to\infty} \int_2^t 3xe^{-4x}\,dx = \lim_{t\to\infty} -\frac{3x}{4}e^{-4x} - \frac{3}{16}e^{-4x}\Big|_2^t$

$$\int 3x e^{-4x}\,dx = -\frac{3x}{4}e^{-4x} - \frac{3}{16}e^{-4x}$$

D	I
$3x$ (+)	e^{-4x}
3 (−)	$-\frac{1}{4}e^{-4x}$
0	$\frac{1}{16}e^{-4x}$

$+\int$

$$= \lim_{t\to\infty}\left[\frac{-3t}{4e^{4t}} - \frac{3}{16}e^{-4t} - \left(\frac{-6}{4}e^{-8} - \frac{3}{16}e^{-8}\right)\right]$$

$$= \frac{6}{4}e^{-8} + \frac{3}{16}e^{-8} = \boxed{\frac{27}{16}e^{-8}}$$

The Integral converges

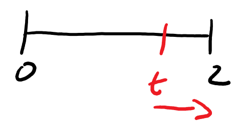
$$\lim_{t\to\infty}\frac{-3t}{4e^{4t}} \overset{LH}{=} \lim_{t\to\infty}\frac{-3}{16e^{4t}} = 0$$

$$\lim_{t\to\infty}\frac{3}{16}e^{-4t} = 0$$

8. $\displaystyle\int_0^5 \frac{5}{(x-2)^3}\,dx$

not continuous at $x=2$
vertical asymptote

$$= \int_0^2 \frac{5}{(x-2)^3}\,dx + \int_2^5 \frac{5}{(x-2)^3}\,dx$$



$$\int_0^2 \frac{5}{(x-2)^3}\,dx + \lim_{t\to 2^-} \int_0^t \frac{5}{(x-2)^3}\,dx$$

$$\int \frac{5}{(x-2)^3}\,dx = \int \frac{5}{u^3}\,du$$

$u = x-2$
$du = dx$

$$= \int 5u^{-3}\,du$$

$$= \frac{5u^{-2}}{-2} = \frac{-5}{2u^2}$$

$$= \frac{-5}{2(x-2)^2}$$

$$= \lim_{t\to 2^-} \left.\frac{-5}{2(x-2)^2}\right|_0^t$$

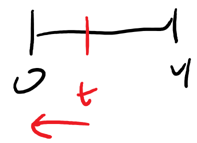
$$= \lim_{t\to 2^-} \left(\underbrace{\frac{-5}{2(t-2)^2}} - \frac{-5}{2(-2)^2}\right)$$

$$= -\infty$$

The Integral diverges (to $-\infty$)

not cont. at $x=0$

9. $\int_0^4 x\ln(x)\,dx = \lim_{t\to 0^+} \int_t^4 x\ln(x)\,dx = \lim_{t\to 0^+} \left(\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right)\Big|_t^4$



$= \lim_{t\to 0^+} \left(8\ln(8) - 4\right) - \left(\frac{t^2}{2}\ln(t) - \frac{t^2}{4}\right)$

$= 8\ln(8) - 4 - (0 - 0)$

$= \underline{\underline{8\ln(8) - 4}}$

$\int x\ln(x)\,dx = \frac{x^2}{2}\ln(x) - \int \frac{x}{2}\,dx$

$= \frac{x^2}{2}\ln(x) - \frac{x^2}{4}$

D		I
$\ln(x)$	+	x
$\frac{1}{x}$	$-\int$	$\frac{x^2}{2}$

L'Hopitals

case $0\cdot(-\infty)$

$\frac{\ln(t)}{\frac{2}{t^2}}$

$\lim_{t\to 0^+} \frac{t^2}{2}\ln(t) = \lim_{t\to 0^+} \frac{\ln(t)}{2t^{-2}} \overset{L'H}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{-4t^{-3}}$

$= \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-4}{t^3}} = \lim_{t\to 0^+} \frac{1}{t}\cdot\frac{t^3}{-4}$

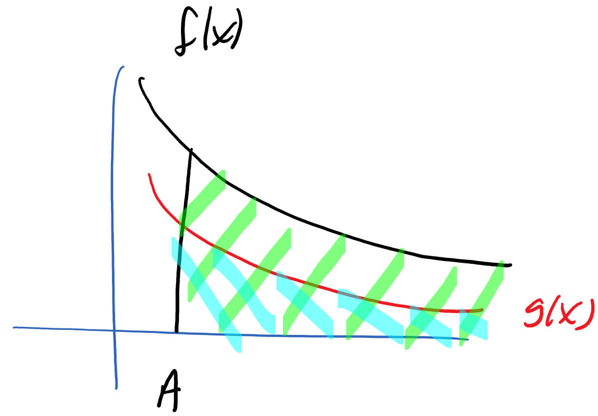
$= \lim_{t\to 0^+} \frac{t^2}{-4} = 0$

Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.



Fact: The $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and diverges if $p \leq 1$.

p- integral

$\int_A^\infty \frac{1}{x^p}\,dx$ with $A > 0$

10. Use the Comparison Theorem to determine if the integral converges or diverges

$$\int_2^\infty \frac{4\sin^2(x)+1}{\sqrt{x}}\,dx = J$$

$$\frac{1}{\sqrt{x}} \le \frac{4\sin^2(x)+1}{\sqrt{x}} \le \frac{5}{\sqrt{x}}$$

$0 \le \sin^2(x) \le 1$

$0 \le 4\sin^2(x) \le 4$

$1 \le 4\sin^2(x)+1 \le 5$

$$\int_2^\infty \frac{1}{\sqrt{x}}\,dx$$

p-integral

$p=\frac{1}{2}$

diverges

$$\int_2^\infty \frac{5}{\sqrt{x}}\,dx$$

p-integrals

$p=\frac{1}{2}$

diverges.

$\int_2^\infty \frac{1}{\sqrt{x}}\,dx$ and Comparison Thm is what Tells us that J diverges

11. Use the Comparison Theorem to determine if the integral converges or diverges

$$\int_{2}^{\infty} \frac{3x}{x^2 + e^{4x}}\, dx$$

will do this one next week.