Section 11.4

The Comparison Test (Strict Comparison): Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(a) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(b) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Limit Comparison Test(LCT): Suppose that $\underbrace{\sum a_n}_{\text{Testing}}$ and $\sum b_n$ are series with positive terms and

Known

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

If L is a number greater than zero, $L > 0$, then both series converge or both series diverge.

Determine if the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{3^n - 1}{5n + 4^n}$

Comparison

$5n + 4^n > 4^n$

$\frac{1}{5n+4^n} < \frac{1}{4^n}$

$\frac{3^n - 1}{5n+4^n} < \frac{3^n - 1}{4^n}$

$3^n - 1 < 3^n$

$\frac{3^n - 1}{4^n} < \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$

$\frac{3^n - 1}{5n+4^n} < \left(\frac{3}{4}\right)^n$

$\sum \left(\frac{3}{4}\right)^n$ geometric $r = \frac{3}{4}$ $|r| < 1$

so it converges

by the comparison test $\sum \frac{3^n - 1}{5n+4^n}$ will conv.

Determine if the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n - 1}{5n + 4^n}$

Try LCT

Looks like

$$\sum \frac{3^n}{4^n} = \sum \left(\frac{3}{4}\right)^n$$

geometric

$r = \frac{3}{4}$

$|r| < 1$ conv.

$$\lim_{n\to\infty} \frac{\frac{3^n - 1}{5n + 4^n}}{\frac{3^n}{4^n}} = \lim_{n\to\infty} \frac{3^n - 1}{5n + 4^n} \cdot \frac{4^n}{3^n}$$

$$= \lim_{n\to\infty} \left(\frac{3^n - 1}{3^n} \cdot \frac{4^n}{5n + 4^n} \right) = 1 \cdot 1 = 1$$

by LCT Both series do same (Both conv or Both div)

So The series conv.

$$\lim_{n\to\infty} \frac{3^n - 1}{3^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{3^n \ln(3)}{3^n \ln(3)} = 1$$

$$\lim_{n\to\infty} \frac{4^n}{5n + 4^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{4^n \ln(4)}{5 + 4^n \ln(4)} \overset{L'H}{=} \lim_{n\to\infty} \frac{4^n (\ln(4))^2}{4^n (\ln(4))^2} = 1$$

2. $\sum_{n=4}^{\infty} \frac{1}{\sqrt[3]{n^2-3}}$

looks like $\sum_{n=4} \frac{1}{\sqrt[3]{n^2}} = \sum \frac{1}{n^{2/3}}$

p-series $p = 2/3 < 1$

diverges

LCT

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n^2-3}}}{\frac{1}{\sqrt[3]{n^2}}} = \lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{\sqrt[3]{n^2-3}} = \lim_{n\to\infty} \sqrt[3]{\boxed{\frac{n^2}{n^2-3}}}$$

$$= \sqrt[3]{1} = 1$$

LCT says Both series do the same

(Both conv or Both div.)

Thus the series diverges

3. $\sum_{n=1}^{\infty} \frac{5+\sin(n)}{5n^3+n+1}$

$4 \le 5+\sin(n) \le 6$

$$\frac{4}{5n^3+n+1} \le \frac{5+\sin(n)}{5n^3+n+1} \le \frac{6}{5n^3+n+1}$$

use LCT with $\underbrace{\sum \frac{1}{n^3}}$ to test. $\sum \frac{6}{5n^3+n+1}$

conv. p-series.

$$\lim_{n\to\infty} \frac{\frac{6}{5n^3+n+1}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{6n^3}{5n^3+n+1} = \frac{6}{5}$$

by LCT $\sum \frac{6}{5n^3+n+1}$ conv.

by the comparison test we get $\sum \frac{5+\sin(n)}{5n^3+n+1}$

converges.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{(2n+5)^4}$

Looks like

$$\sum_{n=1} \frac{\sqrt{n}}{n^4} = \sum_{n=1} \frac{1}{n^{3.5}}$$

p-series $p = 3.5 > 1$

Conv.

Use LCT

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{(2n+5)^4}}{\frac{\sqrt{n}}{n^4}} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{(2n+5)^4} \cdot \frac{n^4}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{n^4}{(2n+5)^4}$$

$$= \lim_{n\to\infty} \sqrt{\boxed{\frac{n+1}{n}}} \cdot \left(\boxed{\frac{n}{2n+5}}\right)^4$$

$$= \underline{\sqrt{1}} \cdot \underline{\left(\frac{1}{2}\right)^4} = \left(\frac{1}{2}\right)^4$$

Since $\sum \frac{\sqrt{n}}{n^4}$ conv. by LCT $\sum \frac{\sqrt{n+1}}{(2n+1)^4}$ conv.

5. $\sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt{n}}$

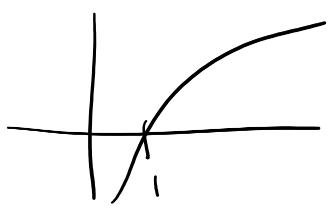
Looks like $\sum \frac{1}{\sqrt{n}}$ div. p-series $p = \frac{1}{2}$

LCT

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \ln(n) = \infty$$

LCT Failed to give information

$\ln(n)$



$1 \leq \ln(n)$

if $n > e = 2.71\ldots$

if $n \geq 3$ then $1 < \ln(n)$

$$\frac{1}{\sqrt{n}} < \frac{\ln(n)}{\sqrt{n}}$$

$\sum \frac{1}{\sqrt{n}}$ p-series $p = \frac{1}{2}$ div.

by comparison test $\sum \frac{\ln(n)}{\sqrt{n}}$ will div.

6. $\displaystyle\sum_{n=4}^{\infty} \frac{4n^3+7n^2}{\sqrt{3n^2+n^{10}+5}}$ $= J$

Looks like $\displaystyle\sum \frac{4n^3}{\sqrt{n^{10}}} = \sum \frac{4n^3}{n^5}$

$$= \sum \frac{4}{n^2}$$

p-series $p = 2$ conv.

LCT with $\displaystyle\sum \frac{4n^3}{\sqrt{n^{10}}}$

$$\lim_{n\to\infty} \frac{\dfrac{4n^3+7n^2}{\sqrt{3n^2+n^{10}+5}}}{\dfrac{4n^3}{\sqrt{n^{10}}}} \quad \lim_{n\to\infty} \frac{4n^3+7n^2}{\sqrt{3n^2+n^{10}+5}} \cdot \frac{\sqrt{n^{10}}}{4n^3}$$

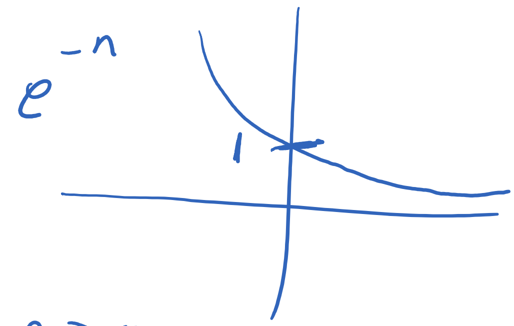
$$= \lim_{n\to\infty} \frac{4n^3+7n^2}{4n^3} \cdot \sqrt{\frac{n^{10}}{n^{10}+3n^2+5}}$$

$$= \frac{4}{4} \cdot \sqrt{1} = 1 \cdot 1 = 1$$

Since $\displaystyle\sum \frac{4}{n^2}$ will converge by LCT

The series J will conv.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{2ne^{-n}}{n^3+4n^2}$



for $n > 0$

$0 < e^{-n} < 1$

$$2ne^{-n} < 2n$$

$$\frac{2ne^{-n}}{n^3+4n^2} < \frac{2n}{n^3+4n^2} \longrightarrow \sum \frac{2n}{n^3+4n^2}$$

new series

Looks like

$$\sum \frac{2n}{n^3} = \sum \frac{2}{n^2}$$

p-series conv.
$p = 2$

use LCT with $\sum \frac{2}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{2n}{n^3+4n^2}}{\frac{2}{n^2}} = \lim_{n\to\infty} \frac{2n}{n^3+4n^2} \cdot \frac{n^2}{2}$$

$$= \lim_{n\to\infty} \frac{2n^3}{2n^3+8n^2} = \frac{2}{2} = 1$$

By LCT $\sum \frac{2n}{n^3+4n^2}$ will conv.

By comparison test we get $\sum \frac{2n e^{-n}}{n^3 + 4/n^2}$ will conv.

8. $\sum_{n=3}^{\infty} \frac{5-\sin^2(n)}{\sqrt{n-1}}$

$0 \le \sin^2(x) \le 1$

$0 \ge -\sin^2(x) \ge -1$

$5 \ge 5-\sin^2(x) \ge 5-1=4$

$4 \le 5-\sin^2(x) \le 5$

$$\frac{4}{\sqrt{n-1}} \le \frac{5-\sin^2(x)}{\sqrt{n-1}} \le \frac{5}{\sqrt{n-1}}$$

Look like $\sum \frac{4}{\sqrt{n}}$ or $\sum \frac{5}{\sqrt{n}}$

p-series $p=\frac{1}{2}$ div.

LCT with $\sum \frac{4}{\sqrt{n}}$ div. $p=\frac{1}{2}$

$$\lim_{n\to\infty} \frac{\frac{4}{\sqrt{n-1}}}{\frac{4}{\sqrt{n}}} = \lim_{n\to\infty} \frac{4}{\sqrt{n-1}} \cdot \frac{\sqrt{n}}{4} = \lim_{n\to\infty} \sqrt{\frac{n}{n-1}}$$

$$= \sqrt{1} = 1$$

by LCT $\sum \frac{4}{\sqrt{n-1}}$ will diverge since $\sum \frac{4}{\sqrt{n}}$ div.

by the comparison test we get $\sum \frac{5-\sin^2(n)}{\sqrt{n-1}}$

will diverge