$(-1)^n \quad (-1)^{n+1}$

Solutions and questions can be found at the link:
https://www.math.tamu.edu/~kahlig/152WIR.html

$a_n = (-1)^{(n-1)} b_n$

The Alternating Series Test (AST): If the alternating series

$\sum_{n=1}^{\infty} \boxed{(-1)^{(n-1)} b_n}$ with $b_n > 0$ satisfies: (1) $\underbrace{b_{n+1} \le b_n \text{ for all } n}_{b_n \text{ dec.}}$ and (2) $\lim_{n\to\infty} b_n = 0$
then the series is convergent.

Test for div.

Alternating Series Estimation Theorem: If $s = \sum_{n=1}^{\infty} (-1)^{(n-1)} b_n$ is the sum of an alternating series that satisfies:

(a) $0 < b_{n+1} \le b_n$ and (b) $\lim_{n\to\infty} b_n = 0$

$\underbrace{b_1 - b_2 + b_3}_{S_3} \underbrace{- b_4 + \cdots}_{R_3}$

then $|R_n| = |s - s_n| \le b_{n+1}$

$= 4$

1. Determine if the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n(n+5)}{n^2+3n}$

$$b_n = \frac{n+5}{n^2+3n}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n+5}{n^2+3n} = 0$$

yes $b_n \to 0$ as $n \to \infty$

$$f(x) = \frac{x+5}{x^2+3x}$$

$$f'(x) = \frac{(x^2+3x)(1) - (x+5)(2x+3)}{(x^2+3x)^2}$$

$$= \frac{x^2+3x - [2x^2+13x+15]}{(x^2+3x)^2}$$

$$= \frac{x^2+3x-2x^2-13x-15}{(x^2+3x)^2}$$

$$= \frac{-x^2-10x-15}{(x^2+3x)^2} < 0$$

$f(x)$ is dec.

by AST we get the series converges.

(b) $\sum_{n=1}^{\infty} (-1)^{2n+1} \cos\left(\frac{\pi}{n}\right)$

$$\sum_{n=1} (-1) \cos\left(\frac{\pi}{n}\right)$$

$$(-1)^{2n+1} = \underbrace{(-1)^{2n}}(-1)^1$$
$$= (+1)(-1)$$
$$= -1$$

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$$

Diverge by test for div.

(c) $\sum_{n=1}^{\infty} \frac{\left(\frac{-1}{3}\right)^n}{n} = \sum_{n=1} \left(\frac{1}{n}\right)\left(\frac{-1}{3}\right)^n = \sum_{n=1} \frac{(-1)^n}{n3^n}$

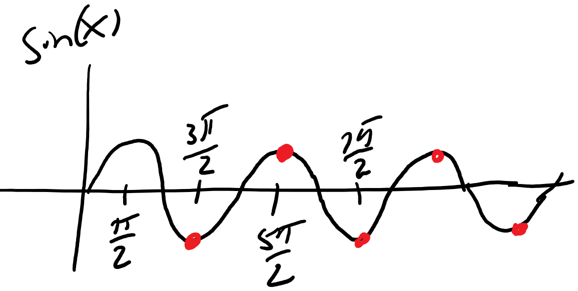
$a_n = (-1)^n b_n$

$\underbrace{b_n = \frac{1}{n3^n}}$ $\leftarrow$ dec.

as $n \to \infty$ $b_n \to 0$

by AST this series will conv.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left((n+\frac{1}{2})\pi\right)}{1+\sqrt{n}}$



$$\sin\left((n+\tfrac{1}{2})\pi\right) = (-1)^n$$

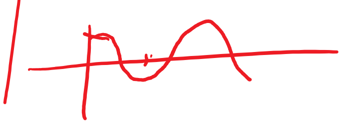
$$b_n = \frac{1}{1+\sqrt{n}}$$

b_n dec

as $n \to \infty$ $b_n \to 0$

by AST the series conv.

$\cos(n\pi) = (-1)^n$



2. Use the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+3)!}$ and the fact it converges by the alternating series test for the following.

(a) Estimate the sum of the series by s_5

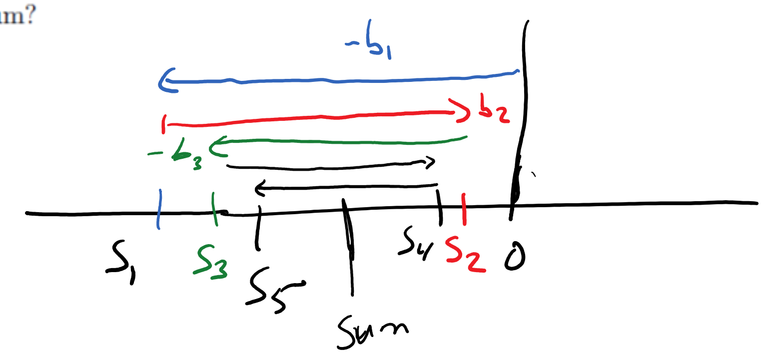
$$\frac{-1}{4!} + \frac{4}{5!} - \frac{9}{6!} + \frac{16}{7!} - \frac{25}{8!}$$

(b) Find an upper bound for the error in the estimate.

$$|R_5| \leq b_6 = \frac{36}{9!}$$

(c) Is the estimate, s_5, more or less than the actual sum?

Less



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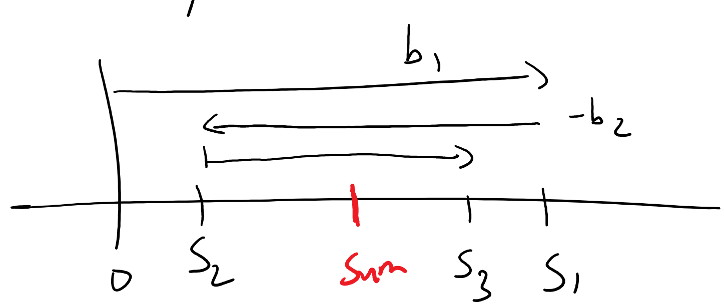
if the first term was positive



3. How many terms of the series do we need to add in order to find the sum so that the |error| < 0.0005?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}$$

$|R_n| < .0005$

$|R_n| \le b_{n+1} < .0005$

$b_n = \frac{1}{n3^n}$

$b_{n+1} = \underbrace{\frac{1}{(n+1)3^{n+1}} < .0005}_{\text{difficult to solve}}$

$b_n = \frac{1}{n3^n}$

$b_1 = \frac{1}{3}$

$b_2 = \frac{1}{2 \cdot 9} = \frac{1}{18}$

$= .05\overline{55}$

$b_3 = \frac{1}{3 \cdot 3^3} = \frac{1}{81}$

$= .0123$

$b_4 = .003086$

$b_5 = .000832$

$\underline{b_6 = .0002386}$

$b_6 < .0005$

Really "b_{n+1}" term

i.e. $n+1 = 6$

$\underline{\underline{n = 5}}$

$\boxed{\text{need 5 terms}}$

original series

new series

Definition: A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ is convergent.

If a series $\sum a_n$ is absolutely convergent then it is also convergent.

Definition: A series $\sum a_n$ is called **conditionally convergent** if the series $\sum |a_n|$ is divergent and the series $\sum a_n$ is convergent.

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4. Determine if the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^4}$$

original series

new series

$$\sum_{n=1} \left| \frac{\cos n}{n^4} \right| = \sum_{n=1} \frac{|\cos n|}{n^4}$$

$$0 \le |\cos n| \le 1$$

$$0 \le \frac{|\cos n|}{n^4} \le \frac{1}{n^4}$$

now look at $\sum \frac{1}{n^4}$ p-series $p=4$ conv.

by the comparison test we get $\sum \frac{|\cos(n)|}{n^4}$ conv.

Thus the original series is Absolutely convergent

Thus it is convergent.

5. Determine if the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{\sqrt{n}}$$

new series

$$\sum_{n=1} \left| \frac{(-1)^n e^{1/n}}{\sqrt{n}} \right| = \sum_{n=1} \frac{e^{1/n}}{\sqrt{n}}$$

use LCT with $\sum \frac{1}{\sqrt{n}}$ p-series $p = \frac{1}{2}$ div.

$$\lim_{n\to\infty} \frac{\frac{e^{1/n}}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{e^{1/n}}{\sqrt{n}} \cdot \frac{\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} e^{1/n} = e^0 = 1$$

Thus $\sum \frac{e^{1/n}}{\sqrt{n}}$ will diverge.

This series is not Abs. conv.

now use AST

$$b_n = \frac{e^{1/n}}{\sqrt{n}} \qquad \lim_{n\to\infty} \frac{e^{1/n}}{\sqrt{n}} = 0$$

is b_n dec?

$$f(x) = \frac{e^{1/x}}{\sqrt{x}}$$

$$f' = \frac{\sqrt{x} \cdot e^{1/x} \cdot \left(\frac{-1}{x^2}\right) - e^{1/x} \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} < 0$$

Thus b_n are dec since $f(x)$ is dec.

by AST $\sum \frac{(-1)^n e^{1/n}}{}$ is conv.

by AST $\sum_{n=1} \frac{(-1)^n e^{1/n}}{\sqrt{n}}$ is conv.

So the series is conditionally conv.

The Ratio Test:

(a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, with $0 \le L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** (and therefore convergent).

(b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: If the limit for the ratio test is 1, then this test fails to give any information. Try something else.

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6. For which series is the Ratio Test inconclusive? (fails to give a definite answer)

(a) $\sum_{n=1}^{\infty} \frac{2n+5}{3n^5-7}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

Inconclusive

(c) $\sum_{n=1}^{\infty} \frac{1}{(-2)^n(n^2+1)}$

(d) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

use Ratio test

$(-2)^n = (-1)^n \underline{2^n}$

$2(n+1)+1 = 2n+2+1 = 2n+3$ | $n+1-1=n$

7. Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-2)^{2n+1}n^4}{3^{n-1}}$

$$a_n = \frac{(-1)^{2n+1}\, 2^{2n+1}\, n^4}{3^{n-1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right|$$

$$a_{n+1} = \frac{(-1)^{2n+3}\, 2^{2n+3}\, (n+1)^4}{3^n}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{2n+3}\, 2^{2n+3}\, (n+1)^4}{3^n}}{\frac{(-1)^{2n+1}\, 2^{2n+1}\, n^4}{3^{n-1}}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{2n+3}\, 2^{2n+3}\, (n+1)^4}{3^n} \cdot \frac{3^{n-1}}{(-1)^{2n+1}\, 2^{2n+1}\, n^4} \right|$$

$$\lim_{n\to\infty} \frac{2^{2n}\, 2^3\, (n+1)^4}{3^n} \cdot \frac{3^n \cdot 3^{-1}}{2^{2n} \cdot 2\, n^4}$$

$$\lim \frac{2^3 \cdot 3^{-1}}{2} \cdot \frac{(n+1)^4}{n^4} = \lim_{n\to\infty} \frac{4}{3}\left(\frac{n+1}{n}\right)^4$$

$$\lim_{n\to\infty} \frac{2 \quad \cdot 3}{2} \cdot \frac{(\ldots)}{n^4} = \lim_{n\to\infty} \frac{1}{3} \left(\frac{\quad}{n} \right)$$

$$= \frac{4}{3} (1)^4 = \frac{4}{3}$$

By the Ratio test the series diverges.

(b) $\sum_{n=1}^{\infty} \frac{n^5}{(-10)^{n+1}}$

$$a_n = \frac{n^5}{(-1)^{n+1} 10^{n+1}} \qquad a_{n+1} = \frac{(n+1)^5}{(-1)^{n+2} 10^{n+2}}$$

$$\lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^5}{(-1)^{n+2} 10^{n+2}} \cdot \frac{(-1)^{n+1} 10^{n+1}}{n^5} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \cdot \boxed{\frac{10^{n+1}}{10^{n+2}}}$$

$$\frac{10^n 10^1}{10^n 10^2} = \frac{10}{10^2} = \frac{1}{10}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^5 \frac{1}{10} = 1^5 \cdot \frac{1}{10} = \frac{1}{10} < 1$$

Thus the series is Abs. Conv. by the Ratio test

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 4!$

$2(n+1) = 2n+2$

(c) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

$$a_n = \frac{(2n)!}{n!\,n!}$$

$$a_{n+1} = \frac{(2n+2)!}{(n+1)!\,(n+1)!}$$

$$a_{n+1} \cdot \frac{1}{a_n}$$

$$\lim_{n\to\infty} \left| \frac{(2n+2)!}{(n+1)!\,(n+1)!} \cdot \frac{n!\,n!}{(2n)!} \right|$$

$$\lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \cdot \frac{n!}{(n+1)\,n!} \cdot \frac{n!}{(n+1)\,n!}$$

$$\lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1} = \frac{4}{1} = 4$$

The series diverges by the Ratio test.

$$3(n+1)+2) = 3n+3+2 = 3n+5$$

(d) $\sum_{n=1}^{\infty} \frac{(-2)^n n!}{5 \cdot 8 \cdot 11 \cdots (3n+2)}$

$$a_n = \frac{(-1)^n 2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots (3n+2)} \qquad a_{n+1} = \frac{(-1)^{n+1} 2^{n+1} (n+1)!}{5 \cdot 8 \cdot 11 \cdot \cdots (3n+2)(3n+5)}$$

$$a_{n+1} \cdot \frac{1}{a_n}$$

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}\, 2^{n+1} (n+1)!}{5 \cdot 8 \cdot 11 \cdot \cdots (3n+2)(3n+5)} \cdot \frac{5 \cdot 8 \cdot 11 \cdot \cdots (3n+2)}{\cancel{(-1)^n}\, 2^n\, n!} \right|$$

$$\lim_{n\to\infty} \frac{2(n+1)\cancel{n!}}{(3n+5)\cancel{n!}} = \lim_{n\to\infty} \frac{2n+2}{3n+5} = \frac{2}{3} < 1$$

by Ratio test the series is Abs. conv.