Math 152 Week in Review: Sections 11.8, 11.9

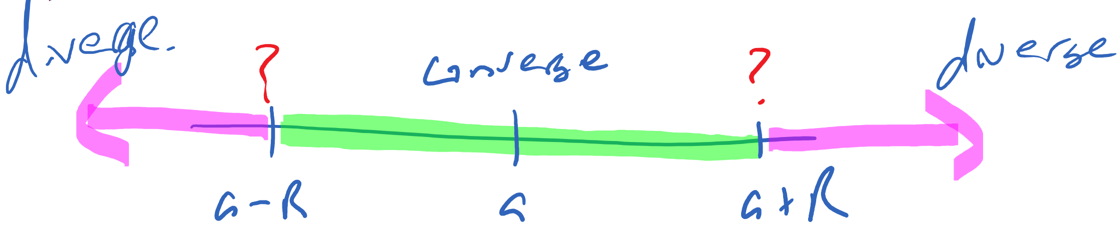
Solutions and questions can be found at the link:
https://www.math.tamu.edu/~kahlig/152WIR.html

For a given power series centered at $x = a$, $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities for convergence.

(i) The series converges only when $x = a$. $R=0$ $I: \{a\}$

(ii) The series converges for all x. $R=\infty$ $I: (-\infty, \infty)$

(iii) The is a positive number R such that the series converges if $\underbrace{|x-a| < R}$ and diverges if $|x-a| > R$.



$R =$ Radius of convergence

$I: (a-R, a+R)$

Test the endpoints

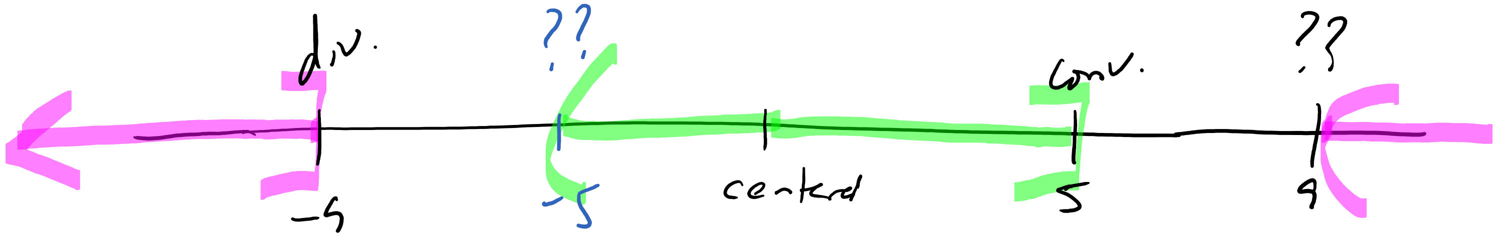
$x = a+R$

$x = a-R$

centered at $a=0$ $(x=0)$

1. Suppose that $\boxed{\sum_{n=0}^{\infty} c_n x^n}$ converges when $x = 5$ but diverges when $x = -9$.

   What can be said about the convergence or divergence of the following series?



Radius of conv. is ≥ 5

Radius of conv. is less than or equal to 9

A) $\sum_{n=0}^{\infty} c_n 10^n$ B) $\sum_{n=0}^{\infty} c_n$ C) $\sum_{n=0}^{\infty} (-1)^n c_n 5^n$ D) $\sum_{n=0}^{\infty} c_n(-2)^n$ E) $\sum_{n=0}^{\infty} c_n 9^n$

$x=10$ $x=1$ $x=-5$ $x=-2$ $x=9$

div. conv. don't know conv. don't know.

$(3n)!$ $\quad 3\boxed{n!} = 3 \cdot n!$

2. Find the interval and the radius of convergence for the following power series.

(a) $\sum_{n=0}^{\infty} \frac{n5^n(x+3)^n}{3n!}$

a_{n+1} $\quad \frac{1}{a_n}$

$$\lim_{n\to\infty} \left| \frac{(n+1)5^{n+1}(x+3)^{n+1}}{\cancel{3}(n+1)!} \cdot \frac{\cancel{3}\cdot n!}{n5^n(x+3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{5^{n+1}}{5^n} \frac{(x+3)^{n+1}}{(x+3)^n}, \frac{n!}{(n+1)!} \right|$$

$(n+1)n!$

$$= \lim_{n\to\infty} \left| \frac{\cancel{n+1}}{n} \cdot 5(x+3) \cdot \frac{1}{\cancel{n+1}} \right| = \lim_{n\to\infty} \left| \frac{5(x+3)}{n} \right| = 0$$

Ratio says conv. (does not depend on X)

ie converge for all x

$R = \infty$ $\quad I: (-\infty, \infty)$

$$(2(n+1))! = (2n+2)! = (2n+2)\cdot(2n+1)(2n)!$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!(5x-1)^n}{n!}$

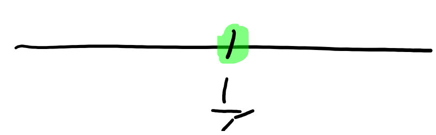
$$\lim_{n\to\infty} \left| \underbrace{\frac{(2n+2)!\,(5x-1)^{n+1}}{(n+1)!}}_{a_{n+1}} \quad \underbrace{\frac{n!}{(2n)!\,(5x-1)^n}}_{\frac{1}{a_n}} \right|$$

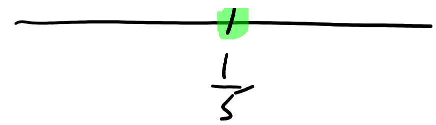
$$\lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)\cancel{(2n)!}\,(5x-1)^{n+1}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{(2n)!}\,(5x-1)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{\overbrace{(2n+2)}^{2(n+1)}(2n+1)\,(5x-1)}{\underbrace{(n+1)}} \right|$$

$$\lim_{n\to\infty} \left| 2\,(2n+1)\,(5x-1) \right| = \begin{cases} \infty & \text{if } x \neq \frac{1}{5} \\ 0 & \text{if } \underbrace{x = \frac{1}{5}}_{\text{series is centered.}} \end{cases}$$

$R = 0$ $\qquad I = \left\{ \frac{1}{5} \right\}$




1/5

(c) $\sum_{n=2}^{\infty} \frac{(2x-5)^n}{n4^n}$

centered at $x = \frac{5}{2}$

$$\lim_{n\to\infty} \left| \frac{(2x-5)^{n+1}}{(n+1)\,4^{n+1}} \cdot \frac{n\,4^n}{(2x-5)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{2x-5}{4} \cdot \frac{n}{n+1} \right| = \left| \frac{2x-5}{4} \right|$$

$$\left| \frac{2x-5}{4} \right| < 1$$

$$|2x-5| < 4$$

$$\left|2\left(x-\tfrac{5}{2}\right)\right| < 4$$

$$2\left|x-\tfrac{5}{2}\right| < 4$$

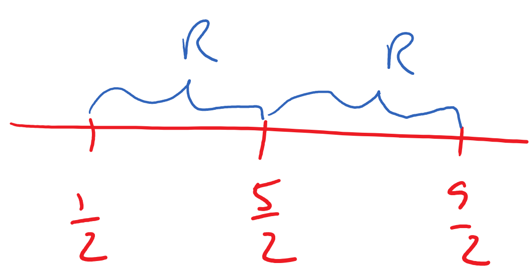
$$\left|x-\tfrac{5}{2}\right| < \boxed{2}$$

$$|x-a| < \boxed{R}$$

$$-4 < 2x-5 < 4$$

$$1 < 2x < 9$$

$$\frac{1}{2} < x < \frac{9}{2}$$



$$R = \frac{9}{2} - \frac{5}{2} = \frac{4}{2} = 2$$

Test the endpoints $\qquad \sum_{n=2} \frac{(2x-5)^n}{n4^n}$

$\boxed{x=\frac{1}{2}}$ $\quad \sum_{n=2} \frac{\left(2\left(\frac{1}{2}\right)-5\right)^n}{n4^n} = \sum_{n=2} \frac{(-4)^n}{n4^n} = \sum_{n=2} \frac{(-1)^n 4^n}{n4^n}$

$= \sum_{n=2} \frac{(-1)^n}{n}$ $\quad$ use AST with $b_n = \frac{1}{n}$

Conv. $\quad$ b_n dec.

as $n \to \infty$ $b_n \to 0$

$\boxed{x=\frac{9}{2}}$ $\quad \sum_{n=2} \frac{\left(2\cdot\frac{9}{2}-5\right)^n}{n4^n} = \sum_{n=2} \frac{4^n}{n4^n} = \sum_{n=2} \frac{1}{n}$

p-series $p=1$

div.

$R=2 \qquad I: \left[\frac{1}{2}, \frac{9}{2}\right)$

Page 6 building blocks

Power Series Building Blocks:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x| < 1 \text{ so } R = 1 \text{ and } I : (-1, 1)$$

$$\underbrace{\frac{1}{1-\square} = \sum_{n=0}^{\infty} \square^n}_{} \text{ if } |\square| < 1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \qquad |x| < 1 \text{ so } R = 1 \text{ and } I : (-1, 1]$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad R = 1 \qquad |x| < 1$$

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3. Find a power series representation for the function. Determine the radius and interval of convergence.

$$g(x) = \frac{5x}{8+27x^3} = \frac{5x}{8\left(1+\frac{27x^3}{8}\right)} = \frac{5x}{8} \cdot \frac{1}{\underbrace{1 - \boxed{\frac{-27x^3}{8}}}}$$

$$= \frac{5x}{8} \sum_{n=0} \left(\frac{-27x^3}{8}\right)^n \qquad \text{if } \left|\frac{-27x^3}{8}\right| < 1$$

$$= \frac{5x}{8} \sum_{n=0} \frac{(-1)^n 27^n x^{3n}}{8^n}$$

$$\frac{5x}{8+27x^3} = \sum_{n=0} \frac{(-1)^n \cdot 5 \cdot 27^n x^{3n+1}}{8^{n+1}}$$

$$|27x^3| < 8$$

$$|x^3| < \frac{8}{27}$$

$$|x| < \sqrt[3]{\frac{8}{27}}$$

$$|x| < \frac{2}{3}$$

$$R = \frac{2}{3}$$

$$I = \left(-\frac{2}{3}, \frac{2}{3}\right)$$

$$\frac{1}{1-\square} = \sum_{n=0} \square^n \text{ if } |\square|<1$$

4. Evaluate the indefinite integral as a power series. What is the radius of convergence?

$$\int \frac{x}{2x^6+3}\,dx$$

$$\frac{x}{2x^6+3} = \frac{x}{3\left(1+\frac{2x^6}{3}\right)} = \frac{x}{3}\cdot\frac{1}{1-\boxed{\frac{-2x^6}{3}}}$$

$$= \frac{x}{3}\cdot\sum_{n=0}\left(\frac{-2x^6}{3}\right)^n$$

if $\left|\frac{-2x^6}{3}\right|<1$

$|2x^6|<3$

$|x^6|<\frac{3}{2}$

$|x|<\sqrt[6]{\frac{3}{2}}$

$$= \frac{x}{3}\sum_{n=0}\frac{(-1)^n 2^n x^{6n}}{3^n}$$

$$\frac{x}{2x^6+3} = \sum_{n=0}\frac{(-1)^n 2^n x^{6n+1}}{3^{n+1}}$$

$R=\sqrt[6]{\frac{3}{2}}$

$$\int\frac{x}{2x^6+3}\,dx = \int\sum_{n=0}\frac{(-1)^n 2^n x^{6n+1}}{3^{n+1}}\,dx$$

$$= C+\sum_{n=0}\frac{(-1)^n 2^n x^{6n+2}}{3^{n+1}\cdot(6n+2)}$$

$$R = \sqrt[6]{\frac{3}{2}}$$

5. Find a power series representation for the function and the Radius of convergence.

$f(x) = \frac{5}{(1-2x)^2}$

$f(x) = \frac{5}{2} g'(x)$

$\frac{1}{1-\square} = \sum_{n=0}^{\infty} \square^n$ if $|\square|<1$

lets start with

$g(x) = \frac{1}{1-2x} = (1-2x)^{-1} = \sum_{n=0} (2x)^n = \underbrace{\sum_{n=0} 2^n x^n}_{1+2x+2^2x^2+\dots}$ if $|2x|<1$

$|x|<\frac{1}{2}$

$R=\frac{1}{2}$

$g'(x) = -1(1-2x)^{-2}(-2)$

$= \frac{2}{(1-2x)^2} = \sum_{n=1} 2^n n x^{n-1}$ $R=\frac{1}{2}$

$f(x) = \frac{5}{2} g'(x) = \frac{5}{2} \sum_{n=1} 2^n n x^{n-1}$

$f(x) = \sum_{n=1} 5 \cdot 2^{n-1} n x^{n-1}$ $R=\frac{1}{2}$

6. Find a power series representation for the function and the Radius of convergence.

$$f(x) = \frac{3x^4}{(1+x)^3}$$

$$f(x) = \frac{3x^4}{2} g''(x)$$

$$g(x) = \frac{1}{1+x} = (1+x)^{-1} = \frac{1}{1-(-x)} = \sum_{n=0} (-x)^n = \sum_{n=0} (-1)^n x^n$$

if $|-x|<1$

$|x|<1$

$R=1$

$$g' = -1(1+x)^{-2} \cdot 1 = \sum_{n=1} (-1)^n n x^{n-1}$$

$R=1$

$$g'' = 2(1+x)^{-3} = \frac{2}{(1+x)^3} = \sum_{n=2} (-1)^n n(n-1) x^{n-2}$$

$R=1$

$$f(x) = \frac{3x^4}{2} g''(x) = \frac{3x^4}{2} \sum_{n=2} (-1)^n n(n-1) x^{n-2}$$

$x^4 x^{n-2} = x^{4+n-2}$

$= x^{n+2}$

$$= \sum_{n=2} \frac{(-1)^n 3n(n-1) x^{n+2}}{2}$$

$R=1$

$$\ln(1+\square) = \sum_{n=0} \frac{(-1)^n \square^{n+1}}{n+1}$$

if $|\square| < 1$

7. Find a power series representation for the function.

$f(x) = \ln(1 + x^2)$

$$= \sum_{n=0} \frac{(-1)^n (x^2)^{n+1}}{n+1}$$

$$= \sum_{n=0} \frac{(-1)^n x^{2n+2}}{n+1}$$

8. Find a power series representation for the function.

$f(x) = \ln(9 - x^3)$

$$\boxed{\ln(1+\square) = \sum_{n=0} \frac{(-1)^n \square^{n+1}}{n+1}}$$

$$f(x) = \ln\left[9\left(1 - \frac{x^3}{9}\right)\right]$$

$$= \ln(9) + \ln\left(1 - \frac{x^3}{9}\right)$$

$$= \ln(9) + \ln\left(1 + \frac{-x^3}{9}\right)$$

$$= \ln(9) + \sum_{n=0} \frac{(-1)^n \left(-\frac{x^3}{9}\right)^{n+1}}{n+1}$$

$$= \ln(9) + \sum_{n=0} \frac{(-1)^n}{n+1} \cdot \frac{(-1)^{n+1} x^{3n+3}}{9^{n+1}}$$

$$= \ln(9) + \sum_{n=0} \frac{\boxed{(-1)^n (-1)^{n+1}} x^{3n+3}}{(n+1) 9^{n+1}}$$

$$= \ln(9) + \sum_{\boxed{n=0}} \frac{-x^{3n+3}}{(n+1) 9^{n+1}}$$

$(-1)^n (-1)^{n+1} = (-1)^{2n+1} = (-1)$

$(-1)^n(-1)^{n+1} = (-1)^{\dots} = (-1)$

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webassign. might have the answer blank like

This

$$\square + \sum_{n=1} \square$$

webassign wants the series shifted

$j = n + 1$

$j - 1 = n$

$$\ln(9) + \sum_{j=1} \frac{-x^{3(j-1)+3}}{j\,9^j}$$

$$\ln(9) + \sum_{j=1} \frac{-x^{3j}}{j\,9^j}$$

$$\boxed{\ln(9) + \sum_{n=1} \frac{-x^{3n}}{n\,9^n}}$$

webassign answer.

$$\arctan(x) = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{2n+1}$$

9. Evaluate the indefinite integral as a power series.

$\int x^2 \arctan(3x^2)\, dx$

$$\arctan(3x^2) = \sum_{n=0} \frac{(-1)^n (3x^2)^{2n+1}}{2n+1}$$

$$= \sum_{n=0} \frac{(-1)^n 3^{2n+1} (x^2)^{2n+1}}{2n+1}$$

$$\arctan(3x^2) = \sum_{n=0} \frac{(-1)^n 3^{2n+1} x^{4n+2}}{2n+1}$$

$$x^2 \arctan(3x^2) = x^2 \sum_{n=0} \frac{(-1)^n 3^{2n+1} x^{4n+2}}{2n+1}$$

$$= \sum_{n=0} \frac{(-1)^n 3^{2n+1} x^{4n+4}}{2n+1}$$

$$\int x^2 \arctan(3x^2)\, dx = \int \sum_{n=0} \frac{(-1)^n 3^{2n+1} x^{4n+4}}{2n+1}\, dx$$

$$= C + \sum_{n=0} \frac{(-1)^n 3^{2n+1} x^{4n+5}}{(2n+1)(4n+5)}$$

Be sure you learn these.

Important Maclaurin series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \qquad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots \qquad R = \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad R = \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \qquad R = \infty$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \qquad -1 < x \leq 1$$