Section 7.8: Improper Integrals.

Improper integrals of Type I

(a) If $\int_a^t f(x)\ dx$ exist for every number $t \geq a$ then $\int_a^\infty f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx$ provided this limit exists (as a finite number).

(b) If $\int_t^a f(x)\ dx$ exist for every number $t \leq a$ then $\int_{-\infty}^a f(x)\ dx = \lim_{t\to-\infty} \int_t^a f(x)\ dx$ provided this limit exists (as a finite number).

The improper integrals in (a) and (b) are called convergent if the limits exists and divergent if the limit does not exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define $\int_{-\infty}^\infty f(x)\ dx = \int_a^\infty f(x)\ dx + \int_{-\infty}^a f(x)\ dx$

Compute these integrals.

A) $\int_2^\infty \frac{1}{(x+3)^{1.5}}\ dx$

B) $\displaystyle\int_0^\infty \frac{\cos(x)}{1+\sin^2(x)}\,dx$

C) $\displaystyle\int_1^\infty \frac{x+1}{(x+3)(x+4)}\,dx$

Fact: The $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and diverges if $p \leq 1$.

Example: For what values of p will these integrals converge?

$\displaystyle\int_1^\infty \frac{5}{x^p}\,dx$ $\displaystyle\int_A^\infty \frac{5}{x^p}\,dx$, where $A > 0$.

Improper integrals of Type II

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$ provided this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$ provided this limit exists (as a finite number).

The improper integrals in (a) and (b) are called convergent if the limits exists and divergent if the limit does not exist.

(c) If f has a discontinuity at c where $a < c < b$, and both $\displaystyle\int_a^c f(x)\,dx$ and $\displaystyle\int_c^b f(x)\,dx$ are convergent, then we define $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Compute these integrals.

A) $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$

B) $\displaystyle\int_{-3}^{1} \frac{1}{x^2}\, dx$

Fact: The $\displaystyle\int_{0}^{1} \frac{1}{x^p}\, dx$ is convergent if $p < 1$ and diverges if $p \geq 1$.

Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$

(a) If $\displaystyle\int_{a}^{\infty} f(x)dx$ is convergent, then $\displaystyle\int_{a}^{\infty} g(x)dx$ is convergent.

(b) If $\displaystyle\int_{a}^{\infty} g(x)dx$ is divergent, then $\displaystyle\int_{a}^{\infty} f(x)dx$ is divergent.

Use the comparison theorem to decide if these integrals converge or diverge.

A) $\displaystyle\int_{2}^{\infty} \frac{1}{\sqrt[3]{x^2 - 1}}\, dx$

B) $\int_{2}^{\infty} \frac{1}{\sqrt[3]{x^4 - 1}}\, dx$

C) $\int_{2}^{\infty} e^{-x^4}\, dx$

D) $\displaystyle\int_1^\infty \frac{3+2\cos(2x)}{x^2}\,dx$

E) $\displaystyle\int_0^\pi \frac{\sin(x)}{\sqrt{x}}\,dx$

Example: Consider the function $f(x) = \dfrac{1}{x}$ is rotated arround the $x - axis$ for the interval $x \geq 2$.

Volume of the solid:

$$V = \int_2^\infty \pi \left(\frac{1}{x}\right)^2 \, dx = \int_2^\infty \frac{\pi}{x^2} \, dx$$

Surface area of the solid:

$$SA = \int_2^\infty 2\pi \frac{1}{x}\sqrt{1 + \frac{1}{x^4}} \, dx = 2\pi \int_2^\infty \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}} \, dx$$

$$SA = 2\pi \int_2^\infty \frac{1}{x}\,\frac{\sqrt{x^4+1}}{x^2} \, dx = 2\pi \int_2^\infty \frac{\sqrt{x^4+1}}{x^3} \, dx$$

Now:

$$\begin{aligned} x^4 &< x^4+1 \\ \sqrt{x^4} &< \sqrt{x^4+1} \\ x^2 &< \sqrt{x^4+1} \\ \frac{x^2}{x^3} &< \frac{\sqrt{x^4+1}}{x^3} \\ \frac{1}{x} &< \frac{\sqrt{x^4+1}}{x^3} \end{aligned}$$