Section 11.10: Taylor and Maclaurin Series

Definition: If a function has a power series representation, then this power series is referred to as the **Taylor series** of the function f at a (or about a or centered at a). If this series is centered at $x = 0$, then this series is given the special name **Maclaurin series**.

Theorem: If $f(x)$ has a power series representation at a, i.e. centered at $x = a$, that is if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with $|x-a| < R$ then its coefficients are given by the formula

$c_n =$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + ...$$

$$f'(x) = c_1 + 2 * c_2(x-a) + 3 * c_3(x-a)^2 + 4 * c_4(x-a)^3 + 5 * c_5(x-a)^4 + ...$$

$$f''(x) = 2 * 1 * c_2 + 3 * 2 * c_3(x-a) + 4 * 3 * c_4(x-a)^2 + 5 * 4 * c_5(x-a)^3 + ...$$

$$f'''(x) = 3 * 2 * 1 * c_3 + 4 * 3 * 2 * c_4(x-a) + 5 * 4 * 3 * c_5(x-a)^2 + 6 * 5 * 4 * c_6(x-a)^3 ...$$

$$f^{(4)}(x) = 4 * 3 * 2 * 1 * c_4 + 5 * 4 * 3 * 2 * c_5(x-a) + 6 * 5 * 4 * 3 * c_6(x-a)^2 + ...$$

$$f^{(5)}(x) = 5 * 4 * 3 * 2 * 1 * c_5 + 6 * 5 * 4 * 3 * 2 * c_6(x-a) + ...$$

Example: Find the Maclaurin series and the radius of convergence for $f(x) = e^{2x}$

Example: Find the Maclaurin series and the radius of convergence for $f(x) = \sin(2x)$

Example: Find the Maclaurin series and the radius of convergence for $f(x) = \cos(2x)$

Important Maclaurin series

Note: These are the only building blocks that you do not have to prove the derivation. Any other "building blocks" used, must be proved.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \qquad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots \qquad R = \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad R = \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \qquad R = \infty$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \qquad |x| \le 1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \qquad -1 < x \le 1$$

Example: Find the Maclaurin series and the radius of convergence for

A) $f(x) = \sin(3x)$

B) $f(x) = x^2 e^{5x}$

Example: Find the Maclaurin series and the radius of convergence for $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

Example: Find the Taylor series of $f(x) = \sin(x)$ at $x = \dfrac{\pi}{6}$

Example: Find the Taylor series of $f(x) = \dfrac{1}{x^2}$ about $a = 3$

Example: Find the Taylor series of $f(x) = \ln(x)$ about $a = 2$

Example: Find the Taylor series of $f(x) = \dfrac{1}{\sqrt{x}}$ about $a = 4$

Example: If $f(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{n4^n}$, find $f^{(48)}(3)$.

Example: Use series to evaluate this integral.

$$\int \frac{e^{x^2} - 1 - x^2}{x} dx$$

Example: Find the sum of these series.

(A) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!}$

(B) $\displaystyle\sum_{n=2}^{\infty} \frac{5^n x^{3n}}{n!}$