Section 11.2: Series

Definition: Given a sequence $\{a_i\}$, we can construct an **infinite series** or **series** by adding the terms of the sequence. $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + ...$

Definition: The n**th partial sum** of a series, denoted s_n, is the sum of the first n-terms.

NOTE: If the index starts at $i = 1$ then

$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$

$s_1 = a_1$
$s_2 = s_1 + a_2 = a_1 + a_2$
$s_3 = s_2 + a_3 = a_1 + a_2 + a_3$
$s_4 = s_3 + a_4 = a_1 + a_2 + a_3 + a_4$
$s_5 = s_4 + a_5 = a_1 + a_2 + a_3 + a_4 + a_5$

Example: Find the s_4 for the series: $\sum_{i=4}^{\infty} \frac{1}{(i-2)^2}$

How To Shift a Series:
Example: Adjust the series $\sum_{i=3}^{\infty} 10\left(\frac{1}{3}\right)^{2i}$ so that the index will now start at i=1.

Definition: Let $\sum_{i=1}^{\infty} a_i$ be a series with s_n being the nth partial sum of this series. If the **sequence of partial sums** $\{s_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$, then we say that the series $\sum_{i=1}^{\infty} a_i$ converges to s or that the series has a sum of s, $\sum_{i=1}^{\infty} a_i = s$. If $\{s_n\}$ does not converge, then the series $\sum_{i=1}^{\infty} a_i$ is said to be divergent.

n	$\mathbf{a_n}$
1	40
2	8
3	$8/5 = 1.6$
4	$8/25 = 0.32$
5	$8/125 = 0.064$
6	$8/625 = 0.0128$
7	$8/\ 3125 = 0.00256$
8	$8/15625 = 0.000512$
9	$8/78125 = 0.0001024$
10	$8/390625 = 0.00002048$

n	$\mathbf{s_n}$
1	40
2	48
3	49.6
4	49.92
5	49.984
6	49.9968
7	49.99936
8	49.999872
9	49.9999744
10	49.99999488

Theorem: If the series $\sum_{i=1}^{\infty} a_i$ is convergent, then $\lim_{i\to\infty} a_i = 0$

<u>Test for Divergence</u>: If $\lim_{i\to\infty} a_i \neq 0$ or DNE, then the series $\sum_{i=1}^{\infty} a_i$ is divergent.

Example: Which of these series DO NOT have a chance at being convergent?

A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

B) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+5}{7-2n}$

C) $\displaystyle\sum_{n=1}^{\infty} \cos(e^{-n})$

Example: The series $\sum_{i=1}^{\infty} a_i$ has a nth partial sum given by s_n. Will the series converge or diverge? Find the formula for the a_n term.

$$s_n = \frac{3n+5}{7-2n}$$

Example: Determine if the Harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, converges or diverges.

Example: The **geometric series** may be defined in a variety of methods.

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ...$$

$$\sum_{n=0}^{\infty} ar^{n} = a + ar + ar^2 + ar^3 + ...$$

$$\sum_{n=7}^{\infty} ar^{n-7} = a + ar + ar^2 + ar^3 + ...$$

Proof of the Geometric Series:
Consider the parital sum of the first n terms.

$$S_n = \sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

Multiply S_n by r to get: $rS_n = ar + ar^2 + ar^3 + ... + ar^n$

Now compute $S_n - rS_n$ and then solve for S_n

$$S_n - rS_n = a - ar^n$$

$$(1-r)S_n = a - ar^n$$

$$S_n = \frac{a - ar^n}{1-r}$$

$$\text{Sum}= \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1-r} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases}$$

Theorem: If $\sum a_n$ and $\sum b_n$ are covergent series, then so are the following series

$\sum ca_n = c\sum a_n$ (where c is a constant)

$\sum(a_n + b_n) = \sum a_n + \sum b_n$

$\sum(a_n - b_n) = \sum a_n - \sum b_n$

Example: Determine if these series are convergent or divergent. If the series is convergent, then give the sum of the series.

A) $1 - \frac{4}{3} + \frac{16}{9} - \frac{64}{27} + ...$

B) $\sum_{i=3}^{\infty} 10\left(\frac{1}{3}\right)^{i-1}$

C) $\sum_{n=0}^{\infty} 7 * 4^{-n}3^{n-1}$

D) $\sum_{i=1}^{\infty} \ln\left(\frac{i}{i+1}\right)$

E) $\displaystyle\sum_{i=3}^{\infty} \left(\frac{1}{i-2} - \frac{1}{i}\right)$

F) $\displaystyle\sum_{i=1}^{\infty} e^{5/(i+1)} - e^{5/i}$

Example: Use a geometric series to express $0.\overline{14}$ as a ratio of integers.

Example: Find the values of x so that $\sum_{n=1}^{\infty}(4x-5)^n$ will converge. Find the sum for those values of x.