Math 152: Cool thing with a power series

I want to compute the value of $\sum_{n=1}^{\infty} \frac{n^2}{6^n}$

If we have $x = \frac{1}{6}$ then we are basically evaluating the power series $\sum_{n=1}^{\infty} n^2 x^n$.

Now lets consider the building blocks that we have. note: the radius of convergence is 1 for all of these.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \sum_{n=1}^{\infty} n x^{n-1}$$

note the change in the index of the series. it now starts at 1.

$$\frac{2}{(1-x)^3} = \frac{d}{dx}\frac{1}{(1-x)^2} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

Now we need to make some adjustments to the building blocks to get x^n in each of the series.

$$\frac{x}{(1-x)^2} = x * \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$$

$$\frac{2x^2}{(1-x)^3} = x^2 * \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^n = \sum_{n=2}^{\infty}(n^2-n)x^n$$

Now the series for $2x^2(1-x)^3$ starts at $n = 2$. We would like it to start at $n = 1$. Looking carefully at the series, $\sum_{n=2}^{\infty} n(n-1)x^n$, notice that if $n = 1$ then the first term would be zero. Thus starting theindex at $n = 1$ or $n = 2$ gives the same series.

i.e. $\frac{2x^2}{(1-x)^3} = \sum_{n=2}^{\infty}(n^2-n)x^n = \sum_{n=1}^{\infty}(n^2-n)x^n$

Now for the "fun". add the fractions

$$\frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty}(n^2-n)x^n + \sum_{n=1}^{\infty} n x^n = \sum_{n=1}^{\infty}(n^2-n)x^n + nx^n = \sum_{n=1}^{\infty} n^2x^n - nx^n + nx^n$$

thus

$$\frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n^2 x^n$$

now just plug in $x = \frac{1}{6}$ into the above formula and compute to get the answer.

$$\sum_{n=1}^{\infty} \frac{n^2}{6^n} = \frac{2(1/6)^2}{(1-1/6)^3} + \frac{1/6}{(1-1/6)^2} = 0.336$$

see this was "cool".