Section 13.3: Arc Length and Curvature

In Cal II, the arc length of a two-dimensional smooth curve that is only traversed once on an interval $I$ was given by

$$L = \int_{I} ds \quad \text{or} \quad L = \int_{I} \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt$$

This can be extended to a space curve. If $\mathbf{r}(t) = (f(t), g(t), h(t))$ on the interval $a \leq t \leq b$, then the length of the curve is given by

$$L = \int_{a}^{b} \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \, dt \quad \text{or} \quad L = \int_{a}^{b} |\mathbf{r}'(t)| \, dt.$$ 

Note: A curve $\mathbf{r}(t)$ is called smooth on an interval if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on the interval. A smooth curve has no sharp corners or cusps, i.e. the tangent vector has continuous movement. The arc length formula holds for smooth and piecewise-smooth curves.

Example: Find the length of the arc for $\mathbf{r}(t) = (3t, 2 \sin(t), 2 \cos(t))$ from the point $(0, 0, 2)$ to $(6\pi, 0, 2)$.

**Definition:** The **arc length function**, $s$, is $s(t) = \int_{a}^{t} |\mathbf{r}'(u)| \, du$.

The arc length $s$ is called the **arc length parameter**.
Example: Find the arc length function for $\mathbf{r}(t) = \langle 1, t^2, t^3 \rangle$ from the point $(1,0,0)$ in the direction of increasing $t$.

Example: Reparametrize the curve $\mathbf{r}(t) = \langle 1 + 2t, 3 + t, -5t \rangle$ with respect to arc length measured from the point where $t = 0$ in the direction of increasing $t$. 
Curvature

**Definition:** The curvature, $\kappa$, of a curve is defined to be the magnitude of the rate of change of the unit tangent vector with respect to the arc length is given by $\kappa = \left| \frac{dT}{ds} \right|$.

**Theorem** The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa = \left| \frac{dT}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{|\mathbf{r}''(s)|}{|\mathbf{r}'(t)|^3}$$

Example: Find the curvature of $\mathbf{r}(t) = \langle -\sqrt{2}\sin t, \cos t, \cos t \rangle$. 

Unit tangent vectors at equally spaced points on $C$
Example: Find the curvature of $\mathbf{r}(t) = \langle 1 + t, 1 - t, 3t^2 \rangle$

Note: The **unit normal vector** is defined as $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ and the **binormal vector** is defined as $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$. $\mathbf{B}(t)$ is also a unit vector.

The plane determined by $\mathbf{N}(t)$ and $\mathbf{B}(t)$ is called the **normal plane**. The plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$ is called the **osculating plane**.