Section 7.8: Additional Problems Solutions

1. The integral is improper since $x = -2$ is not in the domain of the function that we are integrating. Now lets find the anti-derivative. There are two methods to do this.

Method 1: Trig. Sub.

We need $x^2 = 4\sin^2\theta$ so let $x = 2\sin\theta$. This means that $dx = 2\cos\theta d\theta$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,d\theta = \int \frac{2\cos\theta}{\sqrt{4\cos^2\theta}}\,d\theta = \int \frac{2\cos\theta}{2\cos\theta}\,d\theta$$

$$= \int 1\,d\theta = \theta + C = \arcsin\left(\frac{x}{2}\right) + C$$

Method 2: u-sub.

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{4\left(1-\frac{x^2}{4}\right)}}\,dx = \int \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$$

now let $u = \frac{x}{2}$ so we get $du = \frac{1}{2}dx$ or $2du = dx$

$$= \frac{1}{2}\int \frac{2}{\sqrt{1-u^2}}\,du = \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) + C = \arcsin\left(\frac{x}{2}\right) + C$$

Note: The anti-derivative rule used was from section 3.5 in the textbook(covered in Math 151).

$$\int_{-2}^{0} \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{t\to -2^+} \int_{t}^{0} \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{t\to -2^+} \arcsin\left(\frac{x}{2}\right)\Big|_t^0 = \lim_{t\to -2^+} \left[\arcsin(0) = \arcsin\left(\frac{t}{2}\right)\right]$$

$$= 0 - \arcsin(-1) = 0 - \frac{-\pi}{2} = \frac{\pi}{2}$$

Since the result is a number, we know the integral converges. Thus the integral converges to $\frac{\pi}{2}$.

2. This integral is improper since ∞ is one of the limits of the integration. Lets first find the anti-derivative. This is an integration by parts integral.

$$\int \frac{2x}{e^x}\,dx = \int 2xe^{-x}\,dx = -2xe^{-x} - 2e^{-x} + C = \frac{-2x}{e^x} - \frac{2}{e^x} + C$$

$$\int_{3}^{\infty} \frac{2x}{e^x}\,dx = \lim_{t\to\infty} \int_{3}^{t} \frac{2x}{e^x}\,dx = \lim_{t\to\infty} \left(\frac{-2x}{e^x} - \frac{2}{e^x}\right)\Big|_3^t = \lim_{t\to\infty} \left[\frac{-2t}{e^t} - \frac{2}{e^t} - \left(-6e^{-3} - 2e^{-3}\right)\right] = 8e^{-3}$$

since $\lim\limits_{t\to\infty} \frac{2}{e^t} = 0$ and $\lim\limits_{t\to\infty} \frac{-2t}{e^t} \overset{L'H}{=} \lim\limits_{t\to\infty} \frac{-2}{e^t} = 0$

Since the result is a number, we know the integral converges. Thus the integral converges to $8e^{-3}$.

3. This integral is improper since the function is undefined at $x = 1$. Since this value is between the limits of the limits of the integral, we will need to break this into two separate integrals. Lets first find the anti-derivative by a u-sub.

Let $u = x - 1$ then $du = dx$

$$\int \frac{1}{\sqrt[3]{x-1}}\,dx = \int \frac{1}{\sqrt[3]{u}}\,du = \int u^{-1/3}\,du = \frac{3}{2}u^{2/3} + C = \frac{3}{2}(x-1)^{2/3} + C$$

$\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx$ so now we compute each integral.

- $$\int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to1^-}\int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to1^-}\frac{3}{2}(x-1)^{2/3}\Big|_0^t$$
  $$= \lim_{t\to1^-}\left(\frac{3}{2}(t-1)^{2/3} - \frac{3}{2}(-1)^{2/3}\right) = 0 - \frac{3}{2} = \frac{-3}{2}$$
- $$\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to1^+}\int_t^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to1^+}\frac{3}{2}(x-1)^{2/3}\Big|_t^9$$
  $$= \lim_{t\to1^+}\left(\frac{3}{2}(8)^{2/3} - \frac{3}{2}(t-1)^{2/3}\right) = \left(\frac{3}{2}\right)(4) - 0 = 6$$

Since both of the integrals converge we know the original integral will converge.

Answer: $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \frac{-3}{2} + 6 = \frac{9}{2}$

4. We need to find a comparison that can be used to determine if the integral is convergent or divergent.

$$\begin{aligned} -1 \le&\quad \cos(x) &\le 1 \\ -3 \le&\quad 3\cos(x) &\le 3 \\ 2 \le&\quad 3\cos(x)+5 &\le 8 \\ \frac{2}{\sqrt[3]{x}} \le&\quad \frac{3\cos(x)+5}{\sqrt[3]{x}} &\le \frac{8}{\sqrt[3]{x}} \end{aligned}$$

Since we are considering values of x such that $x \ge 2$ we see that all of the terms are positive.

The integrals $\displaystyle\int_2^\infty \frac{2}{\sqrt[3]{x}}\,dx$ and $\displaystyle\int_2^\infty \frac{8}{\sqrt[3]{x}}\,dx$ are both $p-$integrals with $p = \frac{1}{3}$. Both of these integrals will diverge.

Thus the comparison theorem says since $\displaystyle\int_2^\infty \frac{2}{\sqrt[3]{x}}\,dx$ diverges then $\displaystyle\int_2^\infty \frac{3\cos(x)+5}{\sqrt[3]{x}}\,dx$ will also diverge.

5. We need to find a comparison that can be used to determine if the integral is convergent or divergent.

$$\begin{aligned} -1 \leq & \quad \cos(x) & \leq 1 \\ -3 \leq & \quad 3\cos(x) & \leq 3 \\ 2 \leq & \; 3\cos(x)+5 & \leq 8 \\ \frac{2}{x^3} \leq & \; \frac{3\cos(x)+5}{x^3} & \leq \frac{8}{x^3} \end{aligned}$$

Since we are considering values of x such that $x \geq 2$ we see that all of the terms are positive.

The integrals $\int_2^\infty \frac{2}{x^3}\,dx$ and $\int_2^\infty \frac{8}{x^3}\,dx$ are both $p-$integrals with $p = 3$. Both of these integrals will converge

Thus the comparison theorem says since $\int_2^\infty \frac{8}{x^3}$ converges then $\int_2^\infty \frac{3\cos(x)+5}{x^3}\,dx$ will also converge.

6. We need to find a comparison that can be used to determine if the integral is convergent or divergent.

$$\begin{aligned} -1 \leq & \quad \sin(x) & \leq 1 \\ 4 \leq & \; 5+\sin(x) & \leq 6 \\ \frac{4}{x^4} \leq & \; \frac{5+\sin(x)}{x^4} & \leq \frac{6}{x^4} \end{aligned}$$

Since we are considering values of x such that $x \geq 2$ we see that all of the terms are positive.

The integrals $\int_2^\infty \frac{4}{x^4}\,dx$ and $\int_2^\infty \frac{6}{x^4}\,dx$ are both $p-$integrals with $p = 4$. Both of these integrals will converge

Thus the comparison theorem says since $\int_2^\infty \frac{6}{x^4}$ converges then $\int_2^\infty \frac{5+\sin(x)}{x^4}\,dx$ will also converge.

To place a bound on the value of the integral we use the fact that we know

$\int_2^\infty \frac{4}{x^4}\,dx \leq \int_2^\infty \frac{5+\sin(x)}{x^4}\,dx \leq \int_2^\infty \frac{6}{x^4}\,dx$. Now compute the two $p-$integrals.

- $\int_2^\infty \frac{4}{x^4}\,dx = \lim_{t\to\infty} \int_2^t \frac{4}{x^4}\,dx = \lim_{t\to\infty} \left.\frac{-4}{3x^3}\right|_2^t = \lim_{t\to\infty} \left(\frac{-4}{3t^3} - \frac{-4}{3(2)^3}\right) = \frac{1}{6}$
- $\int_2^\infty \frac{6}{x^4}\,dx = \lim_{t\to\infty} \int_2^t \frac{6}{x^4}\,dx = \lim_{t\to\infty} \left.\frac{-6}{3x^3}\right|_2^t = \lim_{t\to\infty} \left(\frac{-6}{3t^3} - \frac{-6}{3(2)^3}\right) = \frac{1}{4}$

Thus $\frac{1}{6} \leq \int_2^\infty \frac{5+\sin(x)}{x^4}\,dx \leq \frac{1}{4}$.

7. First we need to find an anti-derivative. This is done by doing partial fraction decomposition.

$$\frac{24x-4}{(x+2)(3x^2+1)} = \frac{-4}{x+2} + \frac{12x}{3x^2+1}$$

$$\int_1^\infty \frac{24x-4}{(x+2)(3x^2+1)}\,dx = \lim_{t\to\infty}\int_1^t \frac{-4}{x+2} + \frac{12x}{3x^2+1}\,dx = \lim_{t\to\infty}\left(-4\ln|x+2| + 2\ln|3x^2+2|\right)\Big|_1^t$$

$$= \lim_{t\to\infty}\left[\underbrace{-4\ln|t+2| + 2\ln|3t^2+2|}_{\text{L'Hopital case: } \infty-\infty} - \Big(-4\ln|3| + 2\ln|5|\Big)\right]$$

Lets consider the L'Hopital case.

$$\lim_{t\to\infty}\Big(-4\ln|t+2| + 2\ln|3t^2+2|\Big) = \lim_{t\to\infty}\ln\left(\frac{(3t^2+2)^2}{(t+2)^4}\right) = \ln\left(\lim_{t\to\infty}\frac{9t^4+6t^2+1}{t^4+8t^3+24t^2+32t+16}\right)$$

$= \ln(9)$ after repeated use of L'Hopitals Rule on the fraction.

Solution:

$$\int_1^\infty \frac{24x-4}{(x+2)(3x^2+1)}\,dx = \ln(9) + 4\ln(3) - 2\ln(5)$$

8. First lets compute the anti-derivative: $\displaystyle\int \frac{1}{x^p}\,dx = \begin{cases} \ln|x| & \text{if } p = 1 \\ \dfrac{x^{-p+1}}{-p+1} & \text{if } p \neq 1 \end{cases}$

   Now we see that there are two cases depending of the value of p.

Case 1: $P = 1$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(x)\Big|_1^t = \lim_{t\to\infty}\Big(\ln(t) - \ln(1)\Big) = \infty$$

Thus the integral diverges for $p = 1$.

Case 2: $P \neq 1$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg|_1^t = \lim_{t\to\infty}\left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right)$$

In order to finish computing the limit, we need to consider the term t^{-p+1}. Since we know that $P \neq 1$, we either have $-p+1 > 0$ or $-p+1 < 0$.

When $-p+1 > 0$ (equivalently $p < 1$) we see that $\lim\limits_{t\to\infty} t^{-p+1} = \infty$.

When $-p+1 < 0$ (equivalently $p - 1 > 0$ or $p > 1$) and we see that $\lim\limits_{t\to\infty} t^{-p+1} = \lim\limits_{t\to\infty} \dfrac{1}{t^{p-1}} = 0$

This means that the limit is evaluated as the following.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right) = \begin{cases} \infty & \text{if } p < 1 \\ 0 & \text{if } p > 1 \end{cases}$$

From the two cases, we see that the only time this integral has a value, i.e. converges, is when $p > 1$.