Challenge Problem:

(1) Determine if this series is convergent or divergent. If the series is convergent, then give the sum of the series.

$$\sum_{i=1}^{\infty} \frac{1}{i(i+3)}$$

Step 1: Test for divergence.

$$\lim_{i \to \infty} \frac{1}{i(i+3)} = 0$$ so convergence is possible.

Step 2: Since the series is not geometric, look at the partial sums.

To get the n^{th} partial sum we need to decompose the fraction

$$\frac{1}{i(i+3)} = \frac{A}{i} + \frac{B}{i+3} \longrightarrow$$

$1 = A(i+3) + Bi$

if $i=0$ then $A = \frac{1}{3}$

if $i=-3$ then $B = \frac{-1}{3}$

Thus

$$\sum_{i=1}^{\infty} \frac{1}{i(i+3)} = \sum \frac{1/3}{i} - \frac{1/3}{i+3} = \frac{1}{3} \boxed{\sum \frac{1}{i} - \frac{1}{i+3}}$$

now lets look at for the partial sums

partial sum expansion

$$S_n = \sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+3}\right)$$

$$= \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right)$$

$$+ \left(\frac{1}{6} - \frac{1}{9}\right) + \left(\frac{1}{7} - \frac{1}{10}\right) + \cdots + \left(\frac{1}{n-5} - \frac{1}{n-2}\right)$$

$$+ \left(\frac{1}{n-4} - \frac{1}{n-1}\right) + \left(\frac{1}{n-3} - \frac{1}{n}\right) + \left(\frac{1}{n-2} - \frac{1}{n+1}\right)$$

$$+ \left(\frac{1}{n-1} - \frac{1}{n+2}\right) + \left(\frac{1}{n} - \frac{1}{n-3}\right)$$

now cancel terms

$$S_n = \left(1 - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{8}}\right)$$

$$+ \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{9}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{10}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{n-5}} - \cancel{\frac{1}{n-2}}\right)$$

$$+ \left(\cancel{\frac{1}{n-4}} - \cancel{\frac{1}{n-1}}\right) + \left(\cancel{\frac{1}{n-3}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n-2}} - \frac{1}{n+1}\right)$$

$$+ \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+2}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n-3}\right)$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n-3}$$

as $n \to \infty$ $\quad S_n \to 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

Answer: The series converges

and $$\sum_{i=1}^{\infty} \frac{1}{i(i+3)} = \frac{1}{3}\left(\frac{11}{6}\right) = \frac{11}{18}$$