

Green's function asymptotics near the internal edges of spectra of periodic elliptic operators. Spectral edge case.

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Dedicated to the 75th birthday of Professor Eduard Tsekanovskii.

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Abstract

Precise asymptotics known for the Green's function of the Laplace operator have found their analogs for periodic elliptic operators of the second order at and below the bottom of the spectrum. Due to the band-gap structure of the spectra of such operators, the question arises whether similar results can be obtained near or at the edges of spectral gaps. As the result of this work shows, this is possible at a spectral edge when the dimension $d \geq 3$.

1 Introduction

Let

$$L(x, D) = \sum_{j, \ell=1}^d (D_j + b_j(x)) a_{j\ell}(x) (D_\ell + b_\ell(x)) + c(x), \quad (1)$$

be a second-order differential operator in \mathbb{R}^d with smooth real valued coefficients $a_{j\ell}, b_j, c$. Here $D = (D_1, \dots, D_d)$ and $D_j = -i\partial/\partial x_j$.

The operator L is assumed to be **elliptic**, i.e., the matrix $a_{j\ell}$ is symmetric and

$$\sum_{j, \ell=1}^d a_{j\ell} \xi_j \xi_\ell \geq \theta |\xi|^2 \quad (2)$$

for some $\theta > 0$ and any $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$.

We assume that all coefficients are \mathbb{Z}^d -**periodic** where \mathbb{Z}^d is the integer lattice in \mathbb{R}^d , i.e., $a_{jl}(x+n) = a_{jl}(x)$ for all $x \in \mathbb{R}^d, n \in \mathbb{Z}^d$ and similarly for b_j and c .

If the operator is defined as in (1) on the subspace $C_0^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ of compactly supported smooth functions, it is symmetric and essentially self-adjoint [32]. The corresponding self-adjoint operator L has the domain $H^2(\mathbb{R}^d)$.

Most of the conditions above (second order, smoothness of the coefficients, lattice of periods being equal to \mathbb{Z}^d , and sometimes even self-adjointness) can be relaxed, and we postpone a further discussion until Section 4.

In order to formulate the problem addressed in this text and state the results, we introduce some notation and notions from the spectral theory of periodic elliptic operators (see, e.g. [14, 21, 31]).

The spectrum of the defined above (see (1)) operator L in $L^2(\mathbb{R}^d)$ has a **band-gap structure**, i.e., it is the union of a sequence of closed bounded intervals (**bands**) $[\alpha_j, \beta_j] \subset \mathbb{R}, j = 1, 2, \dots$ such that $a_j, b_j \xrightarrow{j \rightarrow \infty} \infty$:

$$\sigma(L) = \bigcup_j [\alpha_j, \beta_j]. \quad (3)$$

The bands can (and do, when $d > 1$ [28, 29, 33]) overlap, but they may leave open intervals in between, called **spectral gaps**. Existence of such gaps, in particular, is crucial for the properties of semi-conductors [3] and photonic crystals [16, 22].

Our interest here is in obtaining asymptotics of the Green's function for the operator L near and at the boundaries of its spectrum. These asymptotics are relevant for many problems in theory of random walks, Martin boundaries, Anderson localization, and others (e.g., [2, 4, 5, 27]). This has been done for the case of the points at and below the bottom of the spectrum of L [4, 27] (see also [35] for description of the results of [4] in discrete case). The resulting asymptotics are similar to the ones for the Laplace operator $-\Delta$. While $\lambda = 0$ is the only boundary point of the spectrum of the Laplace operator, spectral gaps may appear in the periodic case, and thus the question arises of whether one can obtain precise information about asymptotics of the Green's function inside these gaps, or at their edges. The Combes-Thomas estimates (see [5] for their contemporary version) provide the decay estimates for the resolvent in terms of the distance to the spectrum. They, however, do not

reflect the exact asymptotic behavior of the Green's function. It is the goal of this and the next paper to obtain asymptotics or (at least) estimates at and near the spectral gap edges. In this text, only the case of a spectral edge is addressed in dimension $d \geq 3$, while the asymptotics of the Green's function inside a spectral gap (near its edge) will be considered elsewhere. The restriction $d \geq 3$ in the spectral edge case is natural, since at the bottom of the spectrum, the operator is critical in dimensions $d \leq 2$ (see [26, 27, 30]).

Let $W = [0, 1]^d \subset \mathbb{R}^d$ be the unit cube. W is a **fundamental domain** of \mathbb{R}^d with respect to the lattice $\Gamma := \mathbb{Z}^d$, i.e., W and its integer translates cover \mathbb{R}^d and overlap only on the boundaries.

The **dual** (or **reciprocal**) **lattice**¹ of Γ is $\Gamma^* := 2\pi\mathbb{Z}^d$ and its fundamental domain is $B = [-\pi, \pi]^d$. We call B the **Brillouin zone** (although in the solid state physics usually a somewhat different fundamental domain of the reciprocal lattice bears this name [3]).

We also use the notation $\mathbb{T} = \mathbb{R}^d/\mathbb{Z}^d$ and $\mathbb{T}^* = \mathbb{R}^d/2\pi\mathbb{Z}^d$ for the d -dimensional tori corresponding to the lattices \mathbb{Z}^d and $2\pi\mathbb{Z}^d$, respectively.

Definition 1.1. *For any $k \in \mathbb{C}^d$, the subspace $H_k^s = H_k^s(W) \subset H^s(W)$ consists of the restrictions to W of functions $f \in H_{loc}^s(\mathbb{R}^d)$ that satisfy the **Floquet-Bloch condition** that for any $\gamma \in \mathbb{Z}^d$*

$$f(x + \gamma) = e^{ik \cdot \gamma} f(x) \text{ a.e.} \quad (4)$$

Here H^s denotes the standard Sobolev space of order s .

Note that $H_k^0 = L^2(W)$.

We adopt the name **quasimomenta**² for vectors k in (4).

The periodicity of the coefficients of the differential operator $L(x, D)$ preserves condition (4). It thus defines an operator $L(k)$ in $L^2(W)$ with domain $H_k^2(W)$. We could alternatively have defined $L(k)$ as the operator $L(x, D + k)$ in $L^2(\mathbb{T})$ with domain $H^2(\mathbb{T})$. In the first model, $L(k)$ is realized as the k -independent differential expression $L(x, D)$ acting on functions in W with boundary conditions depending on k (which can be identified with sections of a linear bundle over the torus \mathbb{T}), while in the second one, the k -dependent differential expression $L(x, D + k)$ acts on the k -independent domain of periodic functions on W .

¹For a lattice Γ , the elements κ of the dual lattice are characterized by the property that $\kappa \cdot \gamma \in 2\pi\mathbb{Z}$ for any $\gamma \in \Gamma$.

²The name comes from the solid state physics [3, 31].

Notice that condition (4) does not change when k is modified by adding an element of $2\pi\mathbb{Z}^d$. This, when dealing with real values of quasimomentum k , considering vectors $k \in B$ is sufficient.

The hypotheses on L force the spectrum $\sigma(L(k))$ of the operators $L(k)$, $k \in \mathbb{R}^d$, to be discrete. The following classical result (see [3, 21, 31] and the references therein) describes the spectrum of the operator L in $L^2(\mathbb{R}^d)$:

Theorem 1. *The following equality holds between the spectra of L and $L(k)$:*

$$\sigma(L) = \bigcup_{k \in B} \sigma(L(k)). \quad (5)$$

In other words, the spectrum of L is the range of the multiple valued function

$$k \rightarrow \lambda(k) := \sigma(L(k)), k \in B. \quad (6)$$

The graph of this function in $B \times \mathbb{R}$ is called the **dispersion relation** or **dispersion curve**³ for the operator L :

$$\{(k, \lambda) : \lambda \in \sigma(L(k))\}. \quad (7)$$

Taking into account that the operator $L(k)$ is bounded from below, one can label its eigenvalues in non-decreasing order:

$$\lambda_1(k) \leq \lambda_2(k) \leq \dots \quad (8)$$

One thus gets the single-valued, continuous, and piecewise analytic **band functions** $\lambda_j(k)$. The ranges of these functions constitute exactly the bands of the spectrum of L shown in (3).

In this text, we study the Green's function at an edge of the spectrum. Without loss of generality, by shifting the operator by a constant and changing its sign if necessary, we can assume that the edge of interest is 0 and the adjacent spectral band is $[0, a]$ for some $a > 0$ and there is no spectrum for small values of λ below zero. Thus, there is a spectral gap below zero. This means in particular that zero is the minimal value of at least one of the band functions $\lambda_j(k)$.

If zero is the bottom of the entire spectrum, then in many cases (e.g., for periodic non-magnetic Schrödinger operators), it is known that the minimum is attained by a single band $\lambda_j(k)$ function at a single value of the

³Another name of this graph is **Bloch variety** of the operator L , e.g. [21].

quasimomentum k_0 (modulo $2\pi\mathbb{Z}^2$ -periodicity) and is non-degenerate, i.e., the Hessian of $\lambda(k)$ is non-degenerate at k_0 [10–12, 17, 30]⁴. Although this is not necessarily true for other spectral edges, it is expected and commonly assumed in the mathematics and physics literature that it is “generically” true, i.e., it can be achieved by small perturbation of coefficients of the operator. We thus impose the following assumption, under which we will be able to establish our main result:

Assumption A

There exist $k_0 \in B$ and a band function $\lambda_j(k)$ such that:

A1 $\lambda_j(k_0) = 0,$

A2 $|\lambda_i(k)| > \delta > 0$ for $i \neq j,$

A3 k_0 is the only (up to $2\pi\mathbb{Z}^2$ -periodicity) minimum of $\lambda_j,$

A4 $\lambda_j(k)$ is a Morse function near $k_0,$ that is, its Hessian H at k_0 is a positively definite quadratic form.

Thus, in particular,

$$\lambda_j(k) = \frac{1}{2}(k - k_0)^T H(k - k_0) + O(|k - k_0|^3) \quad (9)$$

is the Taylor expansion of λ_j at $k_0.$

The parts A1 and A2 of this assumption are known to be generically true for Schrödinger operators [18]. It is commonly believed, although not proven, that conditions A3 and A4 also are generically satisfied. Our result can be easily reformulated for a somewhat weaker condition than A3. Namely, one can assume

A3' *There are finitely many (up to $2\pi\mathbb{Z}^2$ -periodicity) minima of $\lambda_j.$*

Validity of the non-degeneracy condition A4 is often assumed, for instance in order to define the effective masses [3] or when studying emergence of impurity states under localized perturbations of the periodic medium [6–8].

In what follows, the branch $\lambda_j(k)$ plays a special role, and from now on we use the shorthand notation

$$\lambda(k) := \lambda_j(k). \quad (10)$$

⁴In the non-magnetic Schrödinger case, $k_0 = 0$ [17].

Let us also introduce the notation $\varphi(k_0, x)$ for a \mathbb{Z}^d -periodic function of x such that $\psi(x) = e^{ik_0 \cdot x} \varphi(k_0, x)$ is the (unique up to a constant factor) eigenfunction of $L(k_0)$ with the eigenvalue $\lambda(k_0)$:

$$L(k_0)\psi = \lambda(k_0)\psi. \quad (11)$$

We are ready now to state the main result of this text:

Theorem 2. *Let $d \geq 3$, the operator L satisfy the assumption A, and $R_{-\epsilon} = (L + \epsilon)^{-1}$ for a small $\epsilon > 0$ denote the resolvent of L near the spectral edge $\lambda = 0$ (which exists, due to Assumption A). Then:*

1. For any $\phi, \psi \in L^2_{comp}(\mathbb{R}^d)$,

$$R_{-\epsilon}\phi \cdot \psi \xrightarrow{\epsilon \rightarrow 0} R\phi \cdot \psi$$

for an operator $R : L^2_{comp}(\mathbb{R}^d) \mapsto L^2_{loc}(\mathbb{R}^d)$.

2. The Schwartz kernel $G(x, y)$ of R , which we will simply call **the Green's function of L** , has the following asymptotics when $|x - y| \rightarrow \infty$:

$$\begin{aligned} G(x, y) &= \frac{\pi^{-d/2} \Gamma(\frac{d-2}{2}) e^{i(x-y) \cdot k_0}}{2\sqrt{\det H} |H^{-1/2}(x-y)|^{d-2}} \frac{\varphi(k_0, x) \overline{\varphi(k_0, y)}}{\|\varphi(k_0)\|_{L^2(\mathbb{T})}^2} (1 + O(|x-y|^{-1})) \\ &+ r(x, y), \end{aligned} \quad (12)$$

where $r(x, y) = O(|x - y|^{-N})$ for any $N > 0$, when $|x - y| \rightarrow \infty$.

Here H is the Hessian matrix from (9) and the standard notation $L^2_{comp}(\mathbb{R}^d)$ and $L^2_{loc}(\mathbb{R}^d)$ are used for the spaces of square integrable functions with compact support and locally square integrable functions on \mathbb{R}^d correspondingly.

The idea of the proof is to show that only one branch of the dispersion relation (namely, the one appearing in Assumption A) governs the asymptotics with the crucial term being the Hessian H at the point k_0 . This appears like an analog of homogenization (which is an effect that occurs at the bottom of the spectrum only), and indeed it is a version of it, along with other effects described in [9, 12, 13, 23–25].

We prove Theorem 2 in Section 3 after introducing the tools of Floquet theory [21, 31] in Section 2. Section 4 contains final remarks and is followed by an Acknowledgments Section.

2 A Floquet reduction of the problem

In this section we use Assumption A and reduce the problem of obtaining asymptotics for the Green's function $G(x, y)$ to the problem of finding asymptotics of a scalar integral expression. The scalar integral we obtain resembles the Green's function of the Laplacian.

2.1 Floquet transform

Let $\gamma \in \mathbb{R}^d$ and a function $f(x)$ be defined on \mathbb{R}^d . We then denote by f_γ the γ -shifted version of f : $f_\gamma(x) = f(x + \gamma)$. Assuming that $f(x)$ decays sufficiently fast (e.g., f is compactly supported, although $f \in L^2$ would suffice, since it guarantees the L^2_{loc} -convergence in (14) below), the following transform plays the role of Fourier transform when studying periodic problems (in fact, it is the Fourier transform with respect to the lattice \mathbb{Z}^d of periods):

Definition 2.1. *The **Floquet transform** (also sometimes called **Gelfand transform**)*

$$f(x) \mapsto \hat{f}(k, x) \quad (13)$$

maps a function $f(x)$ of $x \in \mathbb{R}^d$ into a function $\hat{f}(k, x)$ of $(k, x) \in B \times W$, defined as follows:

$$\hat{f}(k) = \hat{f}(k, x) = \sum_{\gamma \in \mathbb{Z}^d} f_\gamma(x) e^{-ik \cdot \gamma} = \sum_{\gamma \in \mathbb{Z}^d} f(x + \gamma) e^{-ik \cdot \gamma}. \quad (14)$$

Notice that $\hat{f}(k, x)$ is $2\pi\mathbb{Z}^d$ -periodic with respect to k (recall that $\Gamma^* := 2\pi\mathbb{Z}^d$ is the dual lattice to $\Gamma := \mathbb{Z}^d$). Thus, it can be naturally interpreted⁵ as a function on the torus \mathbb{T}^* with respect to k . With respect to the x -variable, though, the function is not periodic, but rather satisfies the **cyclic** (or **Floquet**) condition

$$\hat{f}(k, x + \gamma) = e^{ik \cdot \gamma} \hat{f}(k, x) \quad (15)$$

for all $\gamma \in \mathbb{Z}^d$. Thus, the function $\hat{f}(k, x)$ is completely determined by its values on $B \times W$. We usually consider it as a function $\hat{f}(k, \cdot)$ of $k \in B$ with values in a space of functions defined on W .

⁵Formally speaking, this interpretation requires to change the quasimomentum variable k to the **Floquet multiplier** variable $z = (e^{ik_1}, \dots, e^{ik_d})$. We will abuse notation (e.g., in Lemma 2.2), staying with the same variable k .

The next result collects some well known properties of the Floquet transform (e.g., [21]).

Lemma 2.2.

1. The Floquet transform is an isometry (up to a scalar factor) between $L^2(\mathbb{R}^d)$ and $\int_B^\oplus L^2(W) dk = L^2(B, L^2(W))$.

2. Its inversion for any $f \in L^2(\mathbb{R}^d)$ is given by the following two equivalent formulas:

$$f(x) = \int_{\mathbb{T}^*} \hat{f}(k, x) d'k, \quad x \in \mathbb{R}^d \quad (16)$$

and

$$f(x) = \int_{\mathbb{T}^*} \hat{f}(k, x - \gamma) e^{ik \cdot \gamma} d'k, \quad x \in W + \gamma. \quad (17)$$

3. Under the Floquet transform, the action of the operator L in $L^2(\mathbb{R}^d)$ is transformed into

$$\int_B^\oplus L(k),$$

i.e.,

$$\widehat{L}f(k) = L(k)\hat{f}(k)$$

for any $f \in H^2(\mathbb{R}^d)$.

4. Let $\phi(k, x)$ be a function on $\mathbb{R}_k^d \times \mathbb{R}_x^d$ such that it is $2\pi\mathbb{Z}^d$ -periodic with respect to k , belongs to the space H_k^s for each k (see Definition 1.1), and is an infinitely differentiable function of k with values in H_k^s . Then ϕ is the Floquet transform of a function $f \in H^s(\mathbb{R}^d)$. Moreover, for any compact set $K \subset \mathbb{R}^d$ the norm $\|f\|_{H^s(K+\gamma)}$ decays faster than any power of $|\gamma|$. In particular, if $s > d/2$, then

$$|f(x)| \leq C_N(1 + |x|)^{-N} \text{ for any } N.$$

Here $d'k$ denotes the normalized Haar measure on \mathbb{T}^* , or the corresponding Lebesgue measure on B .

2.2 The Floquet reduction

We start by considering the resolvent in the spectral gap:

$$R_{-\epsilon}f = (L + \epsilon)^{-1}f \text{ for } f \in L_{comp}^2(\mathbb{R}^d) \text{ and a small positive } \epsilon.$$

Applying the Floquet transform,

$$\widehat{R_{-\epsilon}f}(k) = (L(k) + \epsilon)^{-1}\hat{f}(k).$$

Given another compactly supported function $\psi \in L_{comp}^2(\mathbb{R}^d)$, we begin our analysis with the sesquilinear form

$$(R_{-\epsilon}f, \psi) = \int_{T^*} \left((L(k) + \epsilon)^{-1}\hat{f}(k), \hat{\psi}(k) \right) d'k. \quad (18)$$

In the following lemma, we prove the first claim of Theorem 2 and introduce an expression for the operator $R = \lim_{\epsilon \rightarrow 0} R_{-\epsilon}$ that is the basis of our study of the asymptotics of the Green's function.

Lemma 2.3. *Let the Assumption A be satisfied. Then the following equality holds:*

$$\lim_{\epsilon \rightarrow 0} (R_{-\epsilon}f, \psi) = \int_{T^*} \left(L(k)^{-1}\hat{f}(k), \hat{\psi}(k) \right) d'k. \quad (19)$$

Thus, the Green's function is the kernel of the operator R defined as follows:

$$\widehat{Rf}(k) = L(k)^{-1}\hat{f}(k). \quad (20)$$

The integral in the right hand side of (19) is absolutely convergent for $f, \psi \in L_{comp}^2$.

Proof. The **complex Bloch variety** of the operator L , denoted by Σ , consists of all complex pairs $(k, \lambda) \in \mathbb{C}^{d+1}$ such that the operator $L(k) - \lambda$ on the torus \mathbb{T} is not invertible. This variety is an analytic subset of co-dimension 1 (e.g., [21]). Moreover, there exist an entire scalar function $h(k, \lambda)$ on \mathbb{C}^{d+1} and an entire operator-valued function $I(k, \lambda)$, such that

1) h vanishes only on (the nonsingular) Σ , has simple zeros at all points of Σ , and

2) outside Σ , the inverse operator $(L(k) - \lambda)^{-1} = h(k, \lambda)^{-1}I(k, \lambda)$ [21, 34].

In particular, for a small $\epsilon > 0$, our operator of interest $(L(k) + \epsilon)^{-1} = h(k, -\epsilon)^{-1}I(k, -\epsilon)$. Due to assumption A, for small $\epsilon > 0$ and quasimomenta

k with small imaginary parts, the function $h(k, -\epsilon)^{-1}$ is equal, up to a non-vanishing analytic factor, to $(\lambda(k) + \epsilon)^{-1}$. Thus, for such values of k and ϵ , one can write the bilinear form $(R_{-\epsilon}f, \phi)$ as

$$(R_{-\epsilon}f, \phi) = \int_{\mathbb{T}^*} \frac{(M(k, \epsilon)\hat{f}(k), \hat{\phi}(k))}{\lambda(k) + \epsilon} d'k. \quad (21)$$

Here $M(k, \epsilon)$ is an operator-function in $L^2(\mathbb{T})$, analytic for small ϵ and nearly real k . The function $\lambda(k)$, due to Assumption A, is a non-negative function with a single simple non-degenerate zero on \mathbb{T} . Also, due to the compactness of support of $f(x)$ and $\psi(x)$ and (14), the functions $\hat{f}(k), \hat{\psi}(k)$ are analytic with respect to k . Thus, for $\epsilon > 0$, the integrand in (21) is nonsingular. Since $d \geq 3$, it is still integrable when $\epsilon = 0$. Hence, a straightforward application of the dominated convergence theorem finishes the proof. \square

Remark 2.4. *Lemma 2.3 could be proven without using the results of [21, 34] concerning the representation of the resolvent $(L(k) - \lambda)^{-1} = h(k, \lambda)^{-1}I(k, \lambda)$ (which requires some regularized infinite determinant techniques). One could instead achieve the same using eigenfunction expansions that are employed below.*

2.3 Singling out the principal term in R

As we have just observed, the Green's function in question is the kernel of the operator R that acts according to (20): for a compactly supported L^2 -function f , the Floquet transform of Rf is given by $\widehat{Rf}(k) = L(k)^{-1}\hat{f}(k)$. In other words,

$$Rf(x) = \int_{\mathbb{T}^*} L(k)^{-1}\hat{f}(k, x)d'k \text{ for } x \in \mathbb{R}^d, \quad (22)$$

or

$$Rf(x + \gamma) = \int_{\mathbb{T}^*} e^{ik \cdot \gamma} L(k)^{-1}\hat{f}(k, x)d'k \text{ for } x \in W, \gamma \in \mathbb{Z}^d. \quad (23)$$

In this section, we will single out the part of the integral in (23) that is responsible for the principal term of the Green's function asymptotics. The first thing to notice is that in order to determine the Schwartz kernel of R , it is sufficient to apply the operator to smooth functions. Thus, we may assume that $f \in C_0^\infty(\mathbb{R}^d)$.

Let us localize the integral around the minimum point k_0 of $\lambda(k)$. In order to do so, we first recall [21, 34] that in a neighborhood of k_0 , there exist functions $\psi(k, x)$ and $\varphi(k, x)$ that are analytic in k and satisfy

1. $\psi(k, x) = e^{ik \cdot x} \varphi(k, x)$,
2. $\varphi(k, x)$ is \mathbb{Z}^d -periodic in x , and
3. $L(k)\psi(k) = \lambda(k)\psi(k)$.

There exists an analytic spectral projector $P(k)$ that projects $L^2(\mathbb{T})$ onto the span of $\psi(k)$.

Let $\nu, \eta \in C_0^\infty(B)$ (here $B = [-\pi, \pi]^d$) be cut-off functions so that $\nu \equiv 1$ near k_0 , $\eta|_{\text{supp } \nu} = 1$, and $\psi(k)$ is well-defined on $\text{supp } \eta$. We can decompose

$$\hat{f} = \nu \hat{f} + (1 - \nu) \hat{f}.$$

The operator $L(k)$ is invertible (in x) whenever k stays away from k_0 . Thus,

$$\hat{u}_g(k, x) = L^{-1}(k)(1 - \nu(k))\hat{f}(k, x)$$

is well-defined and smooth with respect to (k, x) (recall that the operator $L(k)$ is elliptic and depends analytically on k). The smoothness of \hat{u}_g means (according to the statement 4 of Lemma 2.2) that u_g has rapid decay in x (we thus chose the subscript g in u_g to indicate that this is a “good” part of u).

Next, we must solve

$$L(k)\hat{u}(k) = \nu(k)\hat{f}(k).$$

We will look for a solution \hat{u} of the form $\hat{u} = \hat{u}_1 + \hat{u}_2$ where $\hat{u}_1 = P(k)\hat{u}_1$, $\hat{u}_2 = Q(k)\hat{u}_2$, and $Q(k) = I - P(k)$. In this spirit, let $\hat{f}_1 = P(k)\nu(k)\hat{f}$, and $\hat{f}_2 = Q(k)\nu(k)\hat{f}$. Observe that

$$\hat{f}_1 = P(k)\eta(k)\nu(k)\hat{f} = \eta(k)P(k)\nu(k)\hat{f} = \eta(k)\hat{f}_1$$

since η is a function of k and not of x . With the two orthogonal projection operators $P(k)$ and $Q(k)$, the problem of solving $L(k)\hat{u}(k) = \hat{f}(k)$ decomposes into

$$\left(\begin{array}{c|c} L(k)P(k) & 0 \\ \hline 0 & L(k)Q(k) \end{array} \right) \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} = \begin{pmatrix} \hat{f}_1 \\ \hat{f}_2 \end{pmatrix}.$$

In this block-matrix structure, the first column corresponds to a one-dimensional subspace (the range of $P(k)$), while the second one corresponds to the infinite dimensional range of $Q(k)$. Since the operator $L(k)Q(k)$, considered on the range of $Q(k)$, has no spectrum near zero, the operator $(L(k)Q(k))^{-1}$ is well-defined and so is \hat{u}_2 . Additionally, \hat{u}_2 is smooth, so according to Lemma 2.2 again, u_2 decays rapidly as $|x| \rightarrow \infty$.

We know that $\hat{u}_1 = P(k)\hat{u}_1$ and $L(k)\psi(k) = \lambda(k)\psi(k)$. Putting these facts together, we need \hat{u}_1 to satisfy the equality

$$L(k)P(k)\hat{u}_1 = P(k)\hat{f}_1 = \eta(k)P(k)\hat{f}_1,$$

which can be rewritten consecutively in the following forms:

$$\begin{aligned} L(k) \frac{(\hat{u}, \psi(k))_{\mathbb{T}}}{\|\psi(k)\|_{L^2(\mathbb{T})}^2} \psi(k) &= \frac{\eta(k)(\hat{f}_1, \psi(k))_{\mathbb{T}}}{\|\psi(k)\|_{L^2(\mathbb{T})}^2} \psi(k), \\ \frac{(\hat{u}_1, \psi(k))_{\mathbb{T}}}{\|\psi(k)\|_{L^2(\mathbb{T})}^2} L(k)\psi(k) &= \frac{\eta(k)(\hat{f}_1, \psi(k))_{\mathbb{T}}}{\|\psi(k)\|_{L^2(\mathbb{T})}^2} \psi(k), \\ \frac{(\hat{u}_1, \psi(k))_{\mathbb{T}}}{\|\psi(k)\|_{L^2(\mathbb{T})}^2} \lambda(k)\psi(k) &= \frac{\eta(k)(\hat{f}_1, \psi(k))_{\mathbb{T}}}{\|\psi(k)\|_{L^2(\mathbb{T})}^2} \psi(k). \end{aligned}$$

Therefore, it is useful to define

$$\hat{u}_1(k, x) := \frac{\eta(k)(\hat{f}_1(k, \cdot), \psi(k, \cdot))_{\mathbb{T}} \psi(k, x)}{\|\psi(k)\|_{L^2(\mathbb{T})}^2 \lambda(k)}.$$

Applying the inverse Floquet transform, we have for $n \in \mathbb{Z}^d$,

$$u_1(x+n) = \frac{1}{(2\pi)^d} \int_B e^{in \cdot k} \frac{\eta(k)(\hat{f}_1(k, \cdot), \psi(k, \cdot))_{\mathbb{T}} \psi(k, x)}{\|\psi(k)\|_{L^2(\mathbb{T})}^2 \lambda(k)} dk.$$

2.4 From u_1 to a “reduced” Green’s function

We now pass from $u_1(x+n)$ to computing, up to lower order terms at infinity, the Green’s function. Below, we will define a function $G_0(x, y)$ that provides the leading term of the asymptotics of the Green’s function $G(x, y)$. To do so, our plan is to isolate a scalar integral, whose asymptotic behavior is responsible for the asymptotics of the Green’s function.

We introduce $G_0(x, y)$ as follows:

$$u_1(x + n) = \int_{\mathbb{R}^d} G_0(x + n, y) f_1(y) dy.$$

Observe that

$$\begin{aligned} u_1(x + n) &= \frac{1}{(2\pi)^d} \int_B \int_{\mathbb{T}} e^{in \cdot k} \hat{f}_1(k, y) \eta(k) \frac{\overline{\psi(k, y)} \psi(k, x)}{\|\psi(k, \cdot)\|_{\mathbb{T}}^2 \lambda(k)} dy dk \\ &= \frac{1}{(2\pi)^d} \int_B \eta(k) \int_{\mathbb{T}} \sum_{\gamma \in \mathbb{Z}^d} e^{in \cdot k} f_1(y + \gamma) e^{-ik \cdot \gamma} \frac{\overline{\psi(k, y)} \psi(k, x)}{\|\psi(k, \cdot)\|_{\mathbb{T}}^2 \lambda(k)} dy dk \\ &= \frac{1}{(2\pi)^d} \int_B \eta(k) \sum_{\gamma \in \mathbb{Z}^d} \int_{\gamma + \mathbb{T}} e^{in \cdot k} f_1(y) e^{-ik \cdot \gamma} \frac{\overline{\psi(k, y - \gamma)} \psi(k, x)}{\|\psi(k, \cdot)\|_{\mathbb{T}}^2 \lambda(k)} dy dk \end{aligned}$$

We notice that

$$\psi(k, y - \gamma) = e^{ik \cdot (y - \gamma)} \varphi(k, y) = e^{-ik \cdot \gamma} \psi(k, y),$$

so $\overline{\psi(k, y - \gamma)} = e^{ik \cdot \gamma} \overline{\psi(k, y)}$ and

$$u_1(x + n) = \int_{\mathbb{R}^d} f_1(y) \left(\frac{1}{(2\pi)^d} \int_B e^{in \cdot k} \eta(k) \frac{\overline{\psi(k, y)} \psi(k, x)}{\|\psi(k, \cdot)\|_{\mathbb{T}}^2 \lambda(k)} dk \right) dy.$$

Thus, we have

$$G_0(x + n, y) = \frac{1}{(2\pi)^d} \int_B e^{in \cdot k} \eta(k) \frac{\overline{\psi(k, y)} \psi(k, x)}{\|\psi(k, \cdot)\|_{\mathbb{T}}^2 \lambda(k)} dk$$

Note that if $y \in \mathbb{T}$ and $\gamma \in \mathbb{Z}^d$, then

$$G_0(x + n, y + \gamma) = \frac{1}{(2\pi)^d} \int_B e^{i(x+n-(y+\gamma)) \cdot k} \eta(k) \frac{\overline{\varphi(k, y)} \varphi(k, x)}{\|\varphi(k, \cdot)\|_{\mathbb{T}}^2 \lambda(k)} dk$$

or more simply,

$$G_0(x, y) = \frac{1}{(2\pi)^d} \int_B e^{i(x-y) \cdot k} \eta(k) \frac{\overline{\varphi(k, y)} \varphi(k, x)}{\|\varphi(k, \cdot)\|_{\mathbb{T}}^2 \lambda(k)} dk. \quad (24)$$

3 Asymptotics of the Green's function

We start with deducing asymptotics in the reduced case.

3.1 Reduced Green's function G_0

Let

$$\rho(k) := \frac{\overline{\varphi(k, y)}\varphi(k, x)}{\|\varphi(k, \cdot)\|_{\mathbb{T}}^2}$$

Then ρ is a smooth function in k near k_0 , x , and y . Let $\mu_0(k - k_0) = \eta(k)$, so μ_0 is a cutoff function supported near 0. Then

$$G_0(x, y) = \frac{1}{(2\pi)^d} \int_B e^{i(x-y)\cdot k} \frac{\mu_0(k - k_0)\rho(k)}{\lambda(k)} dk. \quad (25)$$

It will be shown in the next subsection that the integral in (25) provides the main term of the asymptotics for the full Green's function $G(x, y)$.

By Taylor expanding around k_0 , we can write

$$\rho(k) = \rho(k_0) + (k - k_0) \cdot \rho_1(k).$$

The integral then becomes

$$\begin{aligned} G_0(x, y) &= \frac{\rho(k_0)}{(2\pi)^d} \int_B e^{i(x-y)\cdot k} \frac{\mu_0(k - k_0)}{\lambda(k)} dk \\ &\quad + \frac{1}{(2\pi)^d} \int_B e^{i(x-y)\cdot k} \frac{\mu_0(k - k_0)(k - k_0) \cdot \rho_1(k)}{\lambda(k)} dk. \end{aligned} \quad (26)$$

We will see that the main term comes from the first integral on the righthand side of (26). It is this integral that we address first.

Assumption A2 means that the matrix $H(\lambda, k_0)$ from Assumption A4 is positive definite. Let $A = H(\lambda, k_0)$. Then

$$\lambda(k) = \frac{1}{2}(k - k_0)^T A (k - k_0) + g_1(k - k_0), \quad (27)$$

where $g_1 = O(|k - k_0|^3)$.

Let $A^{1/2}$ be the positive definite square root of A . Then, after the changes of variables $z = k - k_0$ and $\xi = A^{1/2}z$, we get

$$\begin{aligned} & \int_B e^{i(x-y)\cdot k} \frac{\mu_0(k - k_0)}{\lambda(k)} dk \\ &= e^{i(x-y)\cdot k_0} \int_{\mathbb{R}^d} e^{i(A^{-1/2}(x-y))^T A^{1/2}z} \frac{\mu_0(A^{-1/2}A^{1/2}z)}{\frac{1}{2}(A^{1/2}z)^T A^{1/2}z + g_1(A^{-1/2}A^{1/2}z)} dz \\ &= \frac{2e^{i(x-y)\cdot k_0}}{(\det A)^{1/2}} \int_{\mathbb{R}^d} \frac{e^{iA^{-1/2}(x-y)\cdot \xi}}{\xi \cdot \xi + g(\xi)} \mu(\xi) d\xi. \end{aligned}$$

Here $g = 2g_1 \circ A^{-1/2} = O(|\xi|^3)$ and $\mu = \mu_0 \circ A^{-1/2}$.

Let $x_0 = A^{-1/2}(x - y)$. We have now reduced the problem to finding the asymptotics of the following integral in \mathbb{R}^n :

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{ix_0\cdot \xi}}{\xi \cdot \xi + g(\xi)} \mu(\xi) d\xi, \quad (28)$$

where $g(\xi) = O(|\xi|^3)$. The main goal is to eliminate the extra term $g(\xi)$, after which we have a situation similar to the ordinary Laplacian. To do this, we prove the following proposition:

Proposition 3.1. *Let A , g , and μ be as in (28). If $|x_0| \geq 2$, then*

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{ix_0\cdot \xi} \mu(\xi)}{\xi \cdot \xi + g(\xi)} d\xi = \frac{C_d}{|x_0|^{d-2}} (1 + O(|x_0|^{-1})), \quad |x_0| \rightarrow \infty,$$

where

$$C_d = \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{d/2}}. \quad (29)$$

In the following proof we use the notation c_z for a constant that depends on z only. The value of c_z may change from line to line.

Proof. If the support of μ is small enough, then since $g(\xi) = O(|\xi|^3)$, we have

$$\begin{aligned} \frac{\mu(\xi)}{|\xi|^2 + g(\xi)} &= \frac{\mu(\xi)}{|\xi|^2} \frac{1}{1 + |\xi|^{-2}g(\xi)} \\ &= \frac{\mu(\xi)}{|\xi|^2} + \frac{\mu(\xi)}{|\xi|^2} \sum_{n=1}^{d+1} (-1)^n \frac{g(\xi)^n}{|\xi|^{2n}} + r_1(\xi) \end{aligned}$$

where $r_1(\xi) \in C_0^d(\mathbb{R}^d)$. Next, by Taylor expanding $g(\xi)$, we can write

$$\frac{\mu(\xi)}{|\xi|^2 + g(\xi)} = \frac{\mu(\xi)}{|\xi|^2} + \mu(\xi) \sum_{n=1}^{d+1} \sum_{3n \leq |\alpha| \leq 2n+2+d} \frac{g_{\alpha,n} \xi^\alpha}{|\xi|^{2n+2}} + r(\xi) \quad (30)$$

for some constants $g_{\alpha,n}$ and $r \in C_0^d(\mathbb{R}^d)$.

We start by handling the lower order terms in (30) first, namely the terms with $\xi^\alpha/|\xi|^{2n+2}$ and $r(\xi)$.

3.1.1 Terms of the form $\xi^\alpha/|\xi|^{2n+2}$

Lemma 3.2. *Suppose that $h(x)$ is such that its Fourier transform $\tilde{h}(\xi)$ is a function homogeneous of order -1 . Then*

$$\left| \int_{\mathbb{R}^d} e^{ix_0 \cdot \xi} \tilde{h}(\xi) \mu(\xi) d\xi \right| \leq \frac{C}{|x_0|^{d-1}} \quad (31)$$

for $|x_0| \geq 2$.

Proof. Let us observe that

$$\begin{aligned} \int_{\mathbb{R}^d} e^{ix_0 \cdot \xi} \tilde{h}(\xi) \mu(\xi) d\xi &= \int_{\mathbb{R}^d} h(x) \widetilde{e^{ix_0 \cdot \xi} \mu(x)} dx \\ &= \int_{\mathbb{R}^d} h(x) \tilde{\mu}(x - x_0) dx = \int_{\mathbb{R}^d} h(x - x_0) \tilde{\mu}(x) dx. \end{aligned}$$

Since $\mu \in C_0^\infty(\mathbb{R}^d)$, its inverse Fourier transform $\tilde{\mu}$ is a Schwartz function and hence rapidly decreasing, i.e., there exist constants C_ℓ for $\ell \geq 0$ so that for

$$|D^\ell \tilde{\mu}(x)| \leq \frac{C_\ell}{|x|^\ell}.$$

Next, for $|x_0| \geq 2$, it follows that

$$\begin{aligned} \left| \int_{\mathbb{R}^d} h(x - x_0) \tilde{\mu}(x) dx \right| &\leq \left| \int_{|x-x_0| \leq \frac{1}{2}|x_0|} \frac{1}{|x-x_0|^{d-1}} \frac{C_d}{|x|^d} dx \right| + \left| \int_{|x| < 1} \frac{2^{d-1}}{|x_0|^{d-1}} C_0 dx \right| \\ &\quad + \left| \int_{\substack{|x-x_0| \geq \frac{|x_0|}{2} \\ |x| \geq 1}} \frac{2^{d-1}}{|x_0|^{d-1}} \frac{C_{d+1}}{|x|^{d+1}} dx \right| \leq \frac{c_{d,\mu}}{|x_0|^{d-1}} \end{aligned}$$

for some constant $c_{d,\mu}$ that depends on μ and d . This proves the claim of the lemma. \square

We will now use an integration by parts argument to reduce to the case of degree -1 homogeneity. If $\alpha = (\alpha_1, \dots, \alpha_n)$, then

$$\frac{\partial}{\partial \xi_j} \left(\frac{\xi^\alpha}{|\xi|^\ell} \right) = \frac{-\ell \xi_j \xi^\alpha}{|\xi|^{\ell+2}} + \frac{\alpha_j \xi_1^{\alpha_1} \dots \xi_j^{\alpha_j-1} \dots \xi_n^{\alpha_n}}{|\xi|^\ell},$$

a function that is homogeneous of degree $|\alpha| - 1 - \ell$. Assume that $x_0 = (x_0^1, \dots, x_0^n)$ and $|x_0| \sim |x_0^1|$. Then for $|\alpha| \geq \ell - 1$ and $q := |\alpha| - \ell + 1$ one has

$$\left| \int_{\mathbb{R}^d} e^{ix_0 \cdot \xi} \mu(\xi) \frac{\xi^\alpha}{|\xi|^\ell} d\xi \right| \leq \frac{C}{|x_0^1|^{|\alpha|-\ell+1}} \sum_{j=0}^{|\alpha|-\ell+1} \left| \int_{\mathbb{R}^d} \frac{\partial^j}{\partial \xi_1^j} \left(\frac{\xi^\alpha}{|\xi|^\ell} \right) \frac{\partial^{q-j} \mu(\xi)}{\partial \xi_1^{q-j}} dx \right|.$$

If $j = |\alpha| - \ell + 1$, then we can use (31) with $\tilde{h}(\xi) = \frac{\partial^{|\alpha|-\ell+1}}{\partial \xi_1^{|\alpha|-\ell+1}} \left(\frac{\xi^\alpha}{|\xi|^\ell} \right)$ and compute that

$$\frac{1}{|x_0^1|^{|\alpha|-\ell+1}} \left| \int_{\mathbb{R}^d} \frac{\partial^{|\alpha|-\ell+1}}{\partial \xi_1^{|\alpha|-\ell+1}} \left(\frac{\xi^\alpha}{|\xi|^\ell} \right) \mu(\xi) d\xi \right| \leq C \frac{1}{|x_0|^{|\alpha|-\ell+1+d-1}}.$$

If $j \neq |\alpha| - \ell + 1$, then there exists c so that $c^{-1} \leq |\xi| \leq c$ if $\xi \in \text{supp } \frac{\partial^j \mu}{\partial \xi_1^j}$ since $\mu \in C_0^\infty$ and $\mu \equiv 1$ in a neighborhood of 0. In this case, we can integrate by parts as needed to obtain decay of $O(|x_0|^{-d+1})$ (or of any higher power of $|x_0|$).

Combining our estimates, there exists a constant $C = C(\mu, d, g)$ so that

$$\left| \int_{\mathbb{R}^d} e^{ix_0 \cdot \xi} \mu(x) \sum_{n=1}^{d+1} \sum_{3n \leq |\alpha| \leq 2n+2+d} \frac{g_{\alpha,n} \xi^\alpha}{|\xi|^{2n+2}} d\xi \right| \leq \frac{C}{|x_0|^{d-1}}.$$

3.1.2 Estimate for the $r(\xi)$ term

Since $r(\xi) \in C_0^d(\mathbb{R}^d)$, we may integrate by parts $(d-1)$ times and observe that

$$\left| \int_{\mathbb{R}^d} e^{i|x_0|\xi_1} r(\xi) d\xi \right| = \frac{1}{|x_0|^{d-1}} \int_{\mathbb{R}^d} \left| \frac{\partial^{d-1} r(\xi)}{\partial \xi_1^{d-1}} \right| d\xi \leq \frac{c_d}{|x_0|^{d-1}}.$$

3.1.3 The main term estimate

We will show that for the constant C_d from (29), the following standard asymptotics holds:

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{ix_0 \cdot \xi} \mu(\xi)}{|\xi|^2} d\xi = \frac{C_d}{|x_0|^{d-2}} + O(|x_0|^{-d+1}).$$

Let θ be a radial, smooth approximation of the identity with $\text{supp } \theta \subset B(0, \frac{1}{4})$ and $\int_{\mathbb{R}^d} \theta dx = 1$. Recall that the volume of the unit ball in \mathbb{R}^d is $\omega_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$ and the Newtonian potential is

$$\Psi(x) = \frac{1}{d(d-2)\omega_d} \frac{1}{|x|^{d-2}}.$$

Since Ψ is a harmonic function on $B(x_0, \frac{1}{4})$, it follows from the mean-value property that

$$\Psi * \theta(x_0) = \Psi(x_0).$$

Therefore, denoting by $\beta(x)$ the function whose Fourier transform is $\mu(\xi)$, we get

$$\int_{\mathbb{R}^d} \frac{e^{ix_0 \cdot \xi} \mu(\xi)}{|\xi|^2} d\xi = \int_{\mathbb{R}^d} \frac{e^{ix_0 \cdot \xi} \tilde{\theta}(\xi)}{|\xi|^2} d\xi + \int_{\mathbb{R}^d} \frac{e^{ix_0 \cdot \xi} (\tilde{\beta} - \tilde{\theta})(\xi)}{|\xi|^2} d\xi.$$

The inverse Fourier transform applied to $\tilde{\theta}(\xi)/|\xi|^2$ yields

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{ix_0 \cdot \xi} \tilde{\theta}(\xi)}{|\xi|^2} d\xi = \Psi * \theta(x_0) = \Psi(x_0).$$

Also, since $\tilde{\theta}(0) = \tilde{\beta}(0) = 1$, we can write

$$\frac{(\tilde{\beta} - \tilde{\theta})(\xi)}{|\xi|^2} = \frac{1}{|\xi|^2} \sum_{1 \leq |\alpha| \leq d+1} \xi^\alpha \mu_\alpha(\xi) + \mu'(\xi)$$

where μ_α is rapidly decreasing in $|\xi|$ and $\mu' \in C_0^d(\mathbb{R}^d)$.

Now the argument provided earlier in the proof leads to the desired error estimate

$$\int_{\mathbb{R}^d} e^{ix_0 \cdot \xi} \left(\frac{1}{|\xi|^2} \sum_{1 \leq |\alpha| \leq d+1} \xi^\alpha \mu_\alpha(\xi) + \mu'(\xi) \right) d\xi = O(|x_0|^{1-d}). \quad (32)$$

Finally, since μ_α depends on θ and β , so the O -term in (32) is bounded independently of $|x_0|$. \square

In order to finish the proof of the asymptotics of the reduced Green's function G_0 , it remains to bound the last term in (26):

$$\frac{1}{(2\pi)^d} \int_B e^{i(x-y) \cdot k} \frac{(k - k_0) \cdot \rho_1(k)}{\lambda(k)} \mu_0(k) dk.$$

This integral can be estimated similarly to the technique used for controlling the error terms in Proposition 3.1. Let us use a Taylor expansion in $(k - k_0) \cdot \rho_1(k)$ around k_0 . In the notation from the proof of Proposition 3.1, we have

$$\frac{(k - k_0) \cdot \rho_1(k)}{\lambda(k)} \mu_0(k - k_0) = \mu(\xi) \sum_{n=0}^{d+1} \sum_{3n+1 \leq |\beta| \leq 2n+2+d} c_{\beta,n} \frac{\xi^\beta}{|\xi|^{2n+2}} + R(\xi).$$

The remainder function $R(\xi)$ belongs to $C_0^d(\mathbb{R}^d)$, and as in the proof of Proposition 3.1 it follows that

$$\frac{1}{(2\pi)^d} \int_B e^{ix_0 \cdot \xi} \left(\frac{\mu(\xi)}{|\xi|^2} \sum_{1 \leq |\beta| \leq d} c_\beta \xi^\beta + R(\xi) \right) d\xi = O(|x_0|^{1-d}).$$

If we collect our estimates, we arrive to the following intermediate result:

Theorem 3.3. *Under the conditions of Theorem 2, the following asymptotics holds for the reduced Green's function:*

$$\begin{aligned} G_0(x, y) &= \frac{1}{(2\pi)^d} \int_B e^{i(x-y) \cdot k} \frac{\overline{\varphi(k, y)} \varphi(k, x)}{\|\varphi(k, \cdot)\|_{\mathbb{T}}^2 \lambda(k)} dk \\ &= \frac{\Gamma(\frac{d}{2} - 1) e^{i(x-y) \cdot k_0}}{2\pi^{d/2} (\det A)^{1/2}} \left(\frac{1}{|A^{-1/2}(x-y)|} \right)^{d-2} \frac{\varphi(k_0, x) \overline{\varphi(k_0, y)}}{\|\varphi(k_0)\|_{L^2(\mathbb{T})}^2} (1 + O(|x-y|^{-1})). \end{aligned}$$

3.2 The full Green's function asymptotics

We now need to show that when descending from the true Green's function to the reduced one, we have not changed the principal term of the asymptotics.

Let us recall that the restriction to the reduced Green's function G_0 occurred in Section 2.3 when we dropped the operator term

$$T := \int_{\mathbb{T}^*}^{\oplus} T(k) d'k,$$

where $T(k) = \nu(k)(L(k)Q(k))^{-1} + (1 - \nu(k))(L(k))^{-1}$ is a smooth operator valued function of k . We will show now that the Schwartz kernel of this operator in \mathbb{R}^n decays sufficiently fast to be included into the error term $r(x, y)$ in (12).

To do so, we will need the following auxiliary result:

Lemma 3.4. *For any compactly supported functions $\phi, \theta \in C_0^\infty(\mathbb{R}^d)$, the operator norm in $L^2(\mathbb{R}^d)$ of the operator $\phi_{\gamma'} T \theta_\gamma$, where $g_\gamma(x) := g(x + \gamma)$, satisfies the estimate*

$$\|\phi_{\gamma'} T \theta_\gamma\| = O(|\gamma - \gamma'|^{-N}) \quad \text{for any } N > 0 \text{ when } |\gamma - \gamma'| \rightarrow \infty. \quad (33)$$

Proof. By using unity partitions and shifts, one can assume without loss of generality that $\text{supp } \phi \subset W$ and $\text{supp } \theta \subset W$. Then the Floquet direct integral decomposition of the operator $\phi_{\gamma'} T \theta_\gamma$ is (up to an absolute value 1 exponential factor)

$$\phi(x) \int_{\mathbb{T}^*}^{\oplus} e^{ik \cdot (\gamma - \gamma')} T(k) \theta(x) d^d k,$$

where the functions ϕ and θ are considered as the multiplication operators by them in $L^2(\mathbb{T}^*)$. Then the estimate (33) is just the decay estimate for Fourier coefficients of an infinite differentiable vector (operator) valued function on the torus (in this case, $\phi T(k) \theta$). \square

The operator norm estimate (33) is weaker than the pointwise estimate of the Schwartz kernel r of the remainder in Theorem 2. However, one can bootstrap it from (33) to these pointwise estimates. This can be done in different ways. One would be to use Schauder estimates, while another is to apply the similar reasoning to the Floquet expansion of the operator T in a sufficiently smooth Sobolev space $H^s(\mathbb{R}^d)$ instead of L^2 , which would give corresponding smooth norms estimates on the Schwartz kernel of $\phi_{\gamma'} T \theta_\gamma$.

This finishes the proof of Theorem 2.

4 Final remarks

- In the case when the weaker condition A3' is satisfied, one needs to add the asymptotics coming from all non-degenerate isolated extrema.
- It is easy to reformulate (by spectral shift and, if necessary, changing the sign of the operator) the asymptotics without assuming that $\lambda_0 = 0$ and that this is the upper, rather than lower bank of the gap.
- One might wonder whether one truly needs to have the additive error r in (12) of Theorem 2, not only the multiplicative one. This is indeed

necessary, unless one is dealing with the bottom of the spectrum. The reason is that in other cases the eigenfunctions ϕ in (12) will have zeros, and thus the additive error r cannot be pulled in as a part of the multiplicative one.

- Due to absence of restrictions on the coefficients of the operator L , the result holds for an arbitrary lattice of periods, not necessarily \mathbb{Z}^d .
- The result carries over without any essential changes in the arguments to periodic second order elliptic operators on abelian coverings of compact manifolds (as in [23–25]).
- The smoothness conditions on the coefficients of L are exceedingly severe. One can easily follow the proof to detect that a finite smoothness would suffice. We decided not to pursue this line of study in this article.
- As we have already briefly mentioned, the asymptotics we get has the spirit of an unconventional analog of “homogenization,” which, unlike the classical homogenization, occurs at the internal gap edges, rather than at the bottom of the spectrum. One can find other results that can be interpreted as incarnations of such homogenization in [9, 12, 13, 19, 20, 24, 25].

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