Chapter 2. **Limits and rates of change**
Section 2.2. **The limit of the function**

Definition We write $\boxed{\lim_{x\to a} f(x) = L}$ and say "the limit of $f(x)$, as x approaches a, equals L" if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a but not equal to a.

Definition We write $\boxed{\lim_{x\to a^-} f(x) = L}$ and say the **left-handed limit of** $f(x)$ **as** x **approaches** a **(or the limit of** $f(x)$ **as** x **approaches** a **from the left)**, equals L if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and $x < a$.

Definition We write $\boxed{\lim_{x\to a^+} f(x) = L}$ and say the **right-handed limit of** $f(x)$ **as** x **approaches** a **(or the limit of** $f(x)$ **as** x **approaches** a **from the right)**, equals L if we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and $x > a$.

$$\boxed{\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L}$$

Definition Let f be a function defined on both sides of a, except, possibly at a itself. Then $\boxed{\lim_{x\to a} f(x) = \infty}$ means that the values of $f(x)$ can be made arbitrary large by taking x to be sufficiently close to a but not equal to a.

Definition Let f be a function defined on both sides of a, except, possibly at a itself. Then $\boxed{\lim_{x\to a} f(x) = -\infty}$ means that the values of $f(x)$ can be made arbitrary large negative by taking x to be sufficiently close to a but not equal to a.

Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\boxed{\begin{array}{ccc} \lim_{x\to a} f(x) = \infty & \lim_{x\to a^+} f(x) = \infty & \lim_{x\to a^-} f(x) = \infty \\ \lim_{x\to a} f(x) = -\infty & \lim_{x\to a^+} f(x) = -\infty & \lim_{x\to a^-} f(x) = -\infty \end{array}}$$

Definition We write $\boxed{\lim_{t\to a} \vec{r}(t) = \vec{b}}$ and say "the limit of $\vec{r}(t)$, as t approaches a, equals $\vec{b}$" if we can make vector $\vec{r}(t)$ arbitrary close to $\vec{b}$ by taking t to be sufficiently close to a but not equal to a.

If $\vec{r}(t) =< f(t), g(t) >$, then $\boxed{\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t) \right\rangle}$
provided the limits of the component functions exist.

Section 2.3 **Calculating limits using the limit laws**

Limit laws Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then
1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

7. $\lim_{x\to a} c = c$ 8. $\lim_{x\to a} x = a$

9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer

10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$

Theorem If $f(x) \leq g(x)$ for all x in an open interval that contains a (except possibly at a) and the limits of f an g both exist as x approaches a, then $\boxed{\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)}$

The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\boxed{\lim_{x\to a} g(x) = L}$

Section 2.5 **Continuity**

Definition A function f is *continuous at a number* a if $\boxed{\lim_{x\to a} f(x) = f(a)}$.

If f is not continuous at a, then f has *discontinuity* at a.

If $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, then f has a *jump discontinuity* at a,

if either $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^-} f(x) = \infty$, then f has an *infinity discontinuity* at a and we say line $x = a$ is a *vertical asymptote* of the curve $y = f(x)$

and if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \neq f(a)$, then f has a *removable discontinuity* at a

Definition A function f is *continuous from the right at a number* a if $\boxed{\lim_{x\to a^+} f(x) = f(a)}$,

f is *continuous from the left at a number* a if $\boxed{\lim_{x\to a^-} f(x) = f(a)}$.

Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (At an endpoint of the interval we understand *continuous* to mean *continuous from the right* or *continuous from the left.*)

Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ 2. $f + g$ 3. cf 4. fg 5. $\frac{f}{g}$ if $g(a) \neq 0$

Theorem

(a) Any polynomial is continuous on $(-\infty, \infty)$

(b) Any rational function is continuous on its domain

Theorem If n is a positive even integer, then $f(x) = \sqrt[n]{x}$ is continuous on $[0, \infty)$. If n is a positive odd integer, then f is continuous on $(-\infty, \infty)$.

Theorem If f is continuous at b and $\lim_{x\to a} g(x) = b$, then
$\lim_{x\to a} f(g(x)) = f(b) = f\left(\lim_{x\to a} g(x)\right)$

Theorem If g is continuous at a and f is continuous at $g(a)$, then $(f \circ g)(x) = f(g(x))$ is continuous at a.

The intermediate value theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number strictly between $f(a)$ and $f(b)$. Then there exist a number c in (a, b) such that $f(c) = N$.

Section 2.6 Limits at infinity; horizontal asymptotes

Definition Let f be a function defined on (a, ∞). Then $\boxed{\lim_{x\to\infty} f(x) = L}$ means that we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently large.

Definition Let f be a function defined on $(-\infty, a)$. Then $\boxed{\lim_{x\to-\infty} f(x) = L}$ means that we can make values of $f(x)$ arbitrary close to L by taking x to be sufficiently large negative.

Definition The line $y = L$ is called a **horizontal asymptote of the curve** $y = f(x)$ if either $\boxed{\lim_{x\to\infty} f(x) = L}$ or $\boxed{\lim_{x\to-\infty} f(x) = L}$.

Limit laws Suppose that c is a constant and the limits $\lim_{x\to\pm\infty} f(x)$ and $\lim_{x\to\pm\infty} g(x)$ exist. Then

1. $\lim_{x\to\pm\infty} [f(x) + g(x)] = \lim_{x\to\pm\infty} f(x) + \lim_{x\to\pm\infty} g(x)$
2. $\lim_{x\to\pm\infty} [f(x) - g(x)] = \lim_{x\to\pm\infty} f(x) - \lim_{x\to\pm\infty} g(x)$
3. $\lim_{x\to\pm\infty} cf(x) = c \lim_{x\to\pm\infty} f(x)$
4. $\lim_{x\to\pm\infty} f(x)g(x) = \lim_{x\to\pm\infty} f(x) \cdot \lim_{x\to\pm\infty} g(x)$
5. $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to\pm\infty} f(x)}{\lim_{x\to\pm\infty} g(x)}$ if $\lim_{x\to\pm\infty} g(x) \neq 0$
6. $\lim_{x\to\pm\infty} [f(x)]^n = \left[\lim_{x\to\pm\infty} f(x)\right]^n$ where n is a positive integer
7. $\lim_{x\to\pm\infty} c = c$
11. $\lim_{x\to\pm\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\pm\infty} f(x)}$ where n is a positive integer

Theorem If $r > 0$ is a rational number, then $\boxed{\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to-\infty} \frac{1}{x^r} = 0}$

$$\boxed{\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \begin{cases} \frac{a_n}{b_m}, & \text{if } n = m \\ 0, & \text{if } n < m \\ \infty, & \text{if } n > m \end{cases}}$$

Section 2.7 **Tangents, velocities, and other rates of change**

The tangent line

Let $f(x)$ be a function and suppose a is in domain of f

Definition A tangent line is a line that touches a curve $y = f(x)$ at a point $(a, f(a))$ without cross over.

Problem Find the equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

The equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is $\boxed{y - f(a) = m(x - a)}$, where $\boxed{m = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}}$

Tangent vectors

Let $\vec{r}(t) =< x(t), y(t) >$ be a vector function.

Problem Find a tangent vector to a curve traced by $\vec{r}(t)$ at the point P corresponding to the vector $\vec{r}(a) =< x(a), y(a) >$.

The tangent vector to a curve traced by $\vec{r}(t)$ at the point P corresponding to the vector $\vec{r}(a) =< x(a), y(a) >$ is given by $\boxed{\vec{v} = \lim\limits_{t \to a} \frac{1}{t-a}[\vec{r}(t) - \vec{r}(a)] = \lim\limits_{h \to 0} \frac{1}{h}[\vec{r}(a+h) - \vec{r}(a)]}$

Then the equation of the tangent line to a curve traced by $\vec{r}(t)$ at the point P corresponding to the vector $\vec{r}(a) =< x(a), y(a) >$ is given by $\boxed{\vec{L}(t) = \vec{r}(a) + t\vec{v}}$

Velocity

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement of the object from the origin at time t. Function f is called the **position function** of the object.

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

Then the **velocity** or **instantaneous velocity** at time $t = a$ is $\boxed{v(a) = \lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}}$

Suppose an object moves in the xy-plane in such a way that its position at time t is given by the position vector $\vec{r}(t) =< x(t), y(t) >$.

$$\text{average velocity} = \frac{\vec{r}(a+h) - \vec{r}(h)}{h} = \frac{1}{h}[\vec{r}(a+h) - \vec{r}(a)]$$

The instantaneous velocity $\vec{v}(t)$ at the time $t = a$ is $\boxed{\vec{v}(a) = \lim\limits_{h \to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}}$

The **speed** of a particle is defined to be the magnitude of the velocity vector.

Other rates of change

Suppose y is a quantity that depends on another quantity x or $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is $\boxed{\Delta x = x_2 - x_1}$ and the corresponding change in y is $\boxed{\Delta x = f(x_2) - f(x_1)}$. The difference quotient $\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$ is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$.

The **instantaneous rate of change of y with respect to x** at $x = x_1$ is equal to $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$