Math 151, 510-512, Spring 2008

Review before Test 2.

03/20/2008

Section 3.4 **Derivatives of trigonometric functions**

$$\lim_{x\to 0} \sin x = 0 \qquad \lim_{x\to 0} \cos x = 1$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{\tan x}{x} = 1 \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

Derivatives

$$\frac{d}{dx} \sin x = \cos x \qquad \frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} = \sec^2 x \qquad \frac{d}{dx} \cot x = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x \qquad \frac{d}{dx} \sec x = \sec x \tan x$$

Example

(a) Find $\lim_{x\to 0} \frac{\sin 5x}{\tan 2x}$

(b) Find y' if $y = \sqrt{\csc 2x} + \tan^2(x^2 + 1)$

Section 3.5 **The Chain Rule**

If the derivatives $g'(x)$ and $f'(g(x))$ both exist, and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F'(x)$ exists an is given by the product $\boxed{F(x) = f'(g(x))g'(x)}$

If n is any real number and $u = g(x)$ is differentiable, then
$\boxed{\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)}$

Example Find y' if $y = \sqrt{xe^x + x}$

Section 3.6 **Implicit differentiation**

Example Find $\frac{dy}{dx}$ if $\sin(x + y) = y^2 \tan x$

Two curves are called **orthogonal** if at each point of intersection their tangent lines are perpendicular.

Example Show that the curves $2x^2 + y^2 = 3$ and $x = y^2$ are orthogonal.

Section 3.7 **Derivatives of vector functions**

If $\vec{r}(t) =< x(t), y(t) >$ is a vector function, then
$\boxed{\vec{r}'(t) =< x'(t), y'(t) >}$ if $x'(t)$ and $y'(t)$ exist.

velocity at time t=$\vec{r}'(t) =< x'(t), y'(t) >$

speed at time t=$|\vec{r}'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$

acceleration at time t=$\vec{r}''(t) =< x''(t), y''(t) >$

Example The vector function $\vec{r}(t) =< t, 25t - 5t^2 >$ represents the position of a particle at time t. Find the velocity, speed, and acceleration at $t = 1$.

Section 3.8 **Higher derivatives**

$$\boxed{f''(x) = [f'(x)]'}, \boxed{f'''(x) = [f''(x)]'}, \ldots, \boxed{f^{(n)}(x) = [f^{(n-1)}(x)]'}$$

For the vector function $\vec{r}(t) =< x(t), y(t) >$

$$\boxed{\vec{r}'(t) =< x'(t), y'(t) >}, \boxed{\vec{r}''(t) = [\vec{r}'(t)]' =< x''(t), y''(t) >}, \ldots$$

Example Find y'' if $y = e^{-5x}\cos 3x$

Section 3.9 **Slopes and tangents to parametric curves**

Suppose that curve C is given by the parametric equation $x = x(t)$, $y = y(t)$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Example Find an equation of the tangent to the curve $x(t) = 3t^2 + 1$, $y(t) = 2t^3 + 1$ that pass through the point (4,3).

Section 3.10 **Related rates**

Strategy

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Section 3.11 **Differentials; linear and quadratic approximations**

Definition Let $y = f(x)$, where f is a differentiable function. Then the **differential** dx is an independent variable; that is dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation $\boxed{dy = f'(x)dx}$.

Example The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm. Estimate the maximum error in the calculated surface area.

Suppose that $f(a)$ is a known number and the approximate value is to be calculated for $f(a + \Delta x)$ where Δx is small. Then

$$\boxed{f(a + \Delta x) \approx f(a) + f'(a)\Delta x}$$

Example Use differentials to find an approximate value for $(1.97)^6$.

The approximation $\boxed{f(x) \approx f(a) + f'(a)(x-a)}$ is called the **linear approximation** or **tangent line approximation** of f at a, and the function $\boxed{L(x) = f(a) + f'(a)(x-a)}$ is called the **linearization** of f at a.

The **quadratic approximation** of f near a is
$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$.

Example 8. Find the linear and quadratic approximation to $f(x) = \dfrac{1}{1+x^2}$ near 1.

Section 3.12 **Newton's method**

The **Newton's** or **Newton's-Raphson method** gives *approximations* to the root r of the equation $f(x) = 0$ where f is a differentiable function. First approximation x_1
Second approximation $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

....

nth approximation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

Example 6. Use Newton's method to find $\sqrt[10]{100}$ correct to four decimal places.

Chapter 4. **Inverse functions: exponential, logarithmic, and inverse trigonometric functions**

Section 4.1 **Exponential functions and their derivatives**

An **exponential function** is a function of the form $\boxed{f(x) = a^x}$ where a is a positive constant.
If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is continuous function with domain $(-\infty, \infty)$ and range $(0, \infty)$.

If $0 < a < 1$, $f(x) = a^x$ is decreasing function

if $a > 1$, $f(x) = a^x$ is increasing function

If $a, b > 0$ and x, y are reals, then

$$\boxed{1. a^{x+y} = a^x a^y \quad 2. a^{x-y} = \frac{a^x}{a^y} \quad 3. (a^x)^y = a^{xy} \quad 4. (ab)^x = a^x b^x}$$

If $0 < a < 1$, $\lim_{x \to -\infty} a^x = \infty$, $\lim_{x \to \infty} a^x = 0$

If $a > 1$, $\lim_{x \to -\infty} a^x = 0$, $\lim_{x \to \infty} a^x = \infty$

$$\boxed{\text{e is the number such that } \lim_{h \to 0} \frac{e^h - 1}{h} = 1}$$

e≈2.718281828459045235 36

$$\boxed{(e^x)' = e^x} \qquad \boxed{(e^{g(x)})' = e^{g(x)} g'(x)}$$

Example

(a) Find $\lim_{x \to \infty} \frac{e^{2x}}{e^{2x}+1}$

(b) Find y' if $y = e^{x \cos x}$

Section 4.2 **Inverse functions**

Definition A function f with domain A is called **one-to-one function** if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Horizontal line test A function is one-to-one if and only if no horizontal line intersects its graph more that once.

Definition Let f be one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by $\boxed{f^{-1}(y) = x \Longleftrightarrow f(x) = y}$ for any y in B.

domain of f^{-1} = range of f
range of f^{-1} = domain of f

Let f be one-to-one function with domain A and range B. If $f(a) = b$, then $f^{-1}(b) = a$.

Cancellation equations

$$\boxed{f^{-1}(f(x)) = x \text{ for every } x \in A}$$

$$\boxed{f(f^{-1}(x)) = x \text{ for every } x \in B}$$

How to find the inverse function of a one-to-one function f

1. Write $y = f(x)$
2. Solve this equation for x in terms of y.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

The graph of f^{-1} is obtained by the reflecting the graph f about the line $y = x$.

Theorem If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

Theorem If f is a one-to-one differentiable function with inverse function $g = f^{-1}$ and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and $\boxed{g'(a) = \dfrac{1}{f'(g(a))}}$

Example Find f^{-1} if $f(x) = \frac{1+3x}{5-2x}$.

Example Suppose g is the inverse function of f. Both functions are differentiable. Assuming that $f(1) = 3$, $f(3) = 1$, $f'(1) = 4$, $f'(3) = \frac{1}{2}$. Find $g'(3)$.

Example If $f(x) = x + x^2 + e^x$ and $g(x) = f^{-1}(x)$, find $g'(1)$.