9. Find the area of the surface obtained by rotating the curve $y = x^3$, $0 \le x \le 2$ about the x-axis.

$$S_x = 2\pi \int_0^2 y(x)\sqrt{1+[y'(x)]^2}\,dx$$

$$y'(x) = 3x^2$$

$$= 2\pi \int_0^2 x^3 \sqrt{1+9x^4}\,dx$$

$u = 1+9x^4$

$du = 36x^3\,dx$

$2 \mapsto 1+9(2)^4 = 145$

$0 \mapsto 1+9(0) = 1$

$$= \frac{2\pi}{36} \int_1^{145} \sqrt{u}\,du$$

$$= \frac{\pi}{18}\, \frac{2}{3} u^{3/2} \Big|_1^{145}$$

$$= \boxed{\frac{\pi}{27}\left(145\sqrt{145}-1\right)}$$

10. Find the area of the surface obtained by rotating the curve $x = \sqrt{2y - y^2}$, $0 \le y \le 1$ about the y-axis.

$$S_y = 2\pi \int_0^1 x(y)\sqrt{1 + [x'(y)]^2}\, dy$$

$$x'(y) = \frac{1}{2}(2y - y^2)^{-1/2}(2 - 2y) = \frac{1-y}{\sqrt{2y-y^2}}$$

$$= 2\pi \int_0^1 \sqrt{2y-y^2}\sqrt{1 + \frac{(1-y)^2}{2y-y^2}}\, dy$$

$$= 2\pi \int_0^1 \cancel{\sqrt{2y-y^2}}\sqrt{\frac{\cancel{2y}\cancel{-y^2} + 1 \cancel{-2y} \cancel{+y^2}}{\cancel{2y-y^2}}}\, dy$$

$$= 2\pi \int_0^1 1\, dy$$

$$= \boxed{2\pi}$$

11. Find the following limits

(a) $\lim_{n\to\infty} \frac{\sqrt{n}}{\ln n}$ $= \left|\frac{\infty}{\infty}\right| \overset{L'H.R.}{=} \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2\sqrt{n}}$

$= \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \boxed{\infty}$

(b) $\lim_{n\to\infty} \frac{1-2n^2}{\sqrt[3]{n^6+1}+2n^2}$ $= \lim_{n\to\infty} \frac{n^2\left(\frac{1}{n^2}-2\right)}{\sqrt[3]{n^6\left(1+\frac{1}{n^6}\right)}+2n^2}$

$= \lim_{n\to\infty} \frac{-2n^2}{n^2+2n^2} = \boxed{-\frac{2}{3}}$

(c) $\lim_{n\to\infty} (\sqrt{n+1}-\sqrt{n})$ $= \lim_{n\to\infty} \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$

$= \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \boxed{0}$

12. Find the sum of the series

(a) $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{3^{3n-1}} = \sum_{n=1}^{\infty} \frac{2 \cdot 2^{2n}}{3^{-1} \cdot 3^{3n}} = \frac{2}{1/3} \sum_{n=1}^{\infty} \frac{4^n}{27^n}$

$$= \frac{2}{1/3} \sum_{n=1}^{\infty} \left(\frac{4}{27}\right)^{n-1} \cdot \frac{4}{27} = \frac{2}{1/3} \cdot \frac{4/27}{1-4/27}$$

$$= 6 \cdot \frac{4/27}{23/27} = \boxed{\frac{24}{23}}$$

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n x^2}{n!} = x^2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=2}^{\infty} \frac{x^n}{n!} = e^x - 1 - x$$

$$x^2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} = x^2 \left[e^{-1} - 1 - (-1) \right] = \boxed{x^2 e^{-1}}$$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{6}\right)^{2n} \frac{1}{(2n)!} = \boxed{\frac{\sqrt{3}}{2}}$

$$= \cos\left(\frac{\pi}{6}\right) =$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

13. Which of the following series is convergent?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^{5/7}+1}$

$$\lim_{n\to\infty} \frac{n^2}{n^{5/7}+1} = \lim_{n\to\infty} \frac{n^2}{n^{5/7}\left(1+\frac{1}{n^{5/7}}\right)} = \lim_{n\to\infty} n^{9/7} = \infty$$

$\boxed{\text{diverges}}$ by the Divergence Test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$

$$\frac{0}{3^n} \le \frac{\cos^2 n}{3^n} \le \frac{1}{3^n}$$

$$\frac{\cos^2 n}{3^n} \le \frac{1}{3^n}, \quad \sum_{n=1}^{\infty} \frac{1}{3^n} \text{ converges}$$

(geometric series, $q = \frac{1}{3} < 1$)

By Comparison Test I, $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$ $\boxed{\text{converges}}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Integral Test.

$f(x) = \frac{1}{x(\ln x)^2}$. $\quad \frac{1}{x(\ln x)^2} > 0$ on $[2,\infty)$

$\frac{1}{x(\ln x)^2}$ has discontinuities at $x=0$ and $x=1$.

$\frac{1}{x(\ln x)^2}$ is continuous on $[2,\infty)$.

$$f'(x) = -\frac{1}{x^2(\ln x)^4}\left((\ln x)^2 + 2x\ln x\,\frac{1}{x}\right) = -\frac{2\ln x + (\ln x)^2}{x^2(\ln x)^4}$$

$$= -\frac{2+\ln x}{x^2(\ln x)^3} < 0 \text{ on } [2,\infty).$$

$f(x)$ is decreasing on $[2,\infty)$.

Can do the Integral Test.

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \left| \begin{matrix} u = \ln x \\ du = \frac{1}{x}dx \end{matrix} \right| = \lim_{t\to\infty} \int_2^{t} \frac{dx}{x(\ln x)^2}$$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^2}\,du = \lim_{t\to\infty} \left(-\frac{1}{u}\right)\Bigg|_{\ln 2}^{\ln t} = -\cancel{\lim_{t\to\infty} \frac{1}{\ln t}}^{\,0} + \frac{1}{\ln 2}$$

$$= \frac{1}{\ln 2} < \infty.$$

$\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^2}$ $\boxed{\text{converges,}}$ so does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

14. Which of the following series is absolutely convergent?

(a) $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$

Ratio Test for $a_n = \frac{(-3)^n}{n!}$

$$a_{n+1} = \frac{(-3)^{n+1}}{(n+1)!}$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{(-3)^{n+1}}{(n+1)!}}{\frac{(-3)^n}{n!}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{-3}{n+1} \right| = 0 < 1.$$

$\boxed{\text{converges absolutely}}$

(b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

alternating series, $b_n = \frac{1}{n}$.

$\sum_{n=1}^{\infty} \left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ - diverges (harmonic series).

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.

Alternating series test:

$b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n \longrightarrow b_{n+1} < b_n$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\boxed{\text{converges by AST, but is not absolutely convergent.}}$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{\sqrt{n-2}}$

alternating series, $b_n = \dfrac{n}{\sqrt{n-2}}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{\sqrt{n-2}} = \lim_{n\to\infty}\frac{n}{\sqrt{n}\sqrt{1-\frac{2}{n}}} = \lim_{n\to\infty}\sqrt{n} = \infty$$

$\boxed{\text{diverges by AST}}$

(d) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n}\frac{2^{2n}}{3^{3n}}$

$$\sum_{n=0}^{\infty}\left|(-1)^n\frac{2^{2n}}{3^{3n}}\right| = \sum_{n=0}^{\infty}\frac{2^{2n}}{3^{3n}} = \sum_{n=0}^{\infty}\frac{4^n}{27^n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{4}{27}\right)^n \text{ — converges}$$

(geometric series $r=\frac{4}{27}<1$).

$\boxed{\text{The series is absolutely convergent}}$

15. Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$.

$$c_n = \frac{2^n}{\sqrt{n+3}}$$

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^n}{\sqrt{n+3}}}{\frac{2^{n+1}}{\sqrt{n+4}}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{2^n\sqrt{n+4}}{2^{n+1}\sqrt{n+3}}\right| = \frac{1}{2}$$

Interval of convergence:

$$|x-3| < \frac{1}{2}$$

$$-\frac{1}{2} < x-3 < \frac{1}{2}$$

$$\frac{5}{2} < x < \frac{7}{2}$$

End points: $x = \frac{5}{2}$: $\sum_{n=1}^{\infty} \frac{2^n\left(\frac{5}{2}-3\right)^n}{\sqrt{n+3}}$

$$= \sum_{n=1}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$$ converges by AST.

$x = \frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{2^n\left(\frac{7}{2}-3\right)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}$

diverges (p-series, $p = \frac{1}{2} < 1$).

$\boxed{R = \frac{1}{2}}$ interval of convergence $\left[\frac{5}{2}, \frac{7}{2}\right)$

16. Find the power series representation for the function $f(x) = \ln(3-2x)$ centered at 0.

$$(\ln(3-2x))' = \frac{(-2)}{3-2x} = -\frac{2}{3-2x}$$

$$-\frac{2}{3-2x} = -\frac{2}{3}\cdot\frac{1}{1-\frac{2}{3}x} = -\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}x\right)^n = -\frac{2}{3}\sum_{n=0}^{\infty}\frac{2^n}{3^n}x^n$$

$$= -\sum_{n=0}^{\infty}\frac{2^{n+1}}{3^{n+1}}x^n$$

$$\ln(3-2x) = \int\frac{-2}{3-2x}\,dx = -\int\left(\sum_{n=0}^{\infty}\frac{2^{n+1}}{3^{n+1}}x^n\right)dx$$

$$= -\sum_{n=0}^{\infty}\frac{2^{n+1}}{3^{n+1}}\int x^n\,dx = -\sum_{n=0}^{\infty}\frac{2^{n+1}}{3^{n+1}}\frac{x^{n+1}}{n+1} + C$$

plug $x=0$: $\ln 3 = C$.

$$\ln(3-2x) = \boxed{\ln 3 - \sum_{n=0}^{\infty}\frac{2^{n+1}}{3^{n+1}}\frac{x^{n+1}}{n+1}}$$

17. Find the Taylor series for $f(x) = xe^x$ at $x = 2$.

$$f'(x) = e^x + xe^x = (x+1)e^x$$

$$f''(x) = e^x + (x+1)e^x = (x+2)e^x$$

$$\cdots$$

$$f^{(n)}(x) = (x+n)e^x$$

$$f^{(n)}(2) = (2+n)e^2$$

$$xe^x = \sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n = \boxed{\sum_{n=0}^{\infty}\frac{(2+n)e^2}{n!}(x-2)^n}$$

18. Find the Maclaurin series for $f(x) = x\sin(x^3)$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}$$

$$x\sin(x^3) = x\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{(2n+1)!}}$$

19. Find radius and center of sphere given by the equation $x^2 + y^2 + z^2 = 6x + 4y + 10z$

$$x^2 - 6x + y^2 - 4y + z^2 - 10z = 0$$

$$(x^2 - 6x + 9) + (y^2 - 4y + 4) + (z^2 - 10z + 25) - 9 - 4 - 25 = 0.$$

$$(x-3)^2 + (y-2)^2 + (z-5)^2 = 38$$

$$\boxed{R = \sqrt{38} \quad C(3,2,5)}$$

20. Find the angle between the vectors $\vec{a} = \vec{\imath} + \vec{\jmath} + 2\vec{k}$ and $\vec{b} = 2\vec{\jmath} - 3\vec{k}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{(1)(0) + 1(2) + 2(-3)}{\sqrt{1+1+4}\cdot\sqrt{4+9}} = \frac{-4}{\sqrt{6}\cdot\sqrt{13}}$$

$$= -\frac{4}{\sqrt{78}}$$

$$\boxed{\theta = \cos^{-1}\left(-\frac{4}{\sqrt{78}}\right)}$$

21. Find the directional cosines for the vector $\vec{a} = -2\vec{\imath} + 3\vec{\jmath} + \vec{k}$.

$$|\vec{a}| = \sqrt{4+9+1} = \sqrt{14}$$

$$\boxed{\cos\alpha = -\frac{2}{\sqrt{14}}}$$

$$\boxed{\cos\beta = \frac{3}{\sqrt{14}}}$$

$$\boxed{\cos\gamma = \frac{1}{\sqrt{14}}}$$

22. Find the scalar and the vector projections of the vector $\vec{a} = < 2, -3, 1 >$ onto the vector $\vec{b} = < 1, 6, -2 >$.

$$\text{comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{2(1) + (-3)6 + 1(-2)}{\sqrt{1+36+4}} = \boxed{-\frac{18}{\sqrt{41}}}$$

$$\text{proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} = \boxed{-\frac{18}{41} < 1, 6, -2 >}$$

23. Given vectors $\vec{a} = < -2, 3, 4 >$ and $\vec{b} = < 1, 0, 3 >$. Find $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 4 \\ 1 & 0 & 3 \end{vmatrix} = \vec{i} \begin{vmatrix} 3 & 4 \\ 0 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} -2 & 4 \\ 1 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} -2 & 3 \\ 1 & 0 \end{vmatrix}$$

$$= 9\vec{i} - \vec{j}(-6-4) + \vec{k}(0-3)$$

$$= \boxed{9\vec{i} + 10\vec{j} - 3\vec{k}}$$

24. Find the volume of the parallelepiped determined by vectors $\vec{a} = < 1, 0, 6 >$, $\vec{b} = < 2, 3, -8 >$, and $\vec{c} = < 8, -5, 6 >$.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 0 & 6 \\ 2 & 3 & -8 \\ 8 & -5 & 6 \end{vmatrix} = 1 \begin{vmatrix} 3 & -8 \\ -5 & 6 \end{vmatrix} - 0 \begin{vmatrix} 2 & -8 \\ 8 & 6 \end{vmatrix} + 6 \begin{vmatrix} 2 & 3 \\ 8 & -5 \end{vmatrix}$$

$$= (18 - 40) + 6(-10 - 24) = -22 - 204 = -226$$

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = |-226| = \boxed{226}$$

25. Represent the point with Cartesian coordinates $(2\sqrt{3}, -2)$ in terms of polar coordinates.

$x = 2\sqrt{3}$, $y = -2$ IV quadrant.

$r^2 = x^2 + y^2 = 4(3) + 4 = 16$, $r = 4$.

$\tan\theta = \frac{y}{x} = -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$

$\theta = \frac{11\pi}{6}$.

$$\boxed{\left(4, \frac{11\pi}{6}\right) \text{ or } \left(-4, \frac{5\pi}{6}\right)}$$

26. Sketch the curve $r = \sin 5\theta$.