Bifurcations of vector fields on the two-sphere

Based on a joint works with Yu. Ilyashenko, I. Schurov, and N. Solodovnikov

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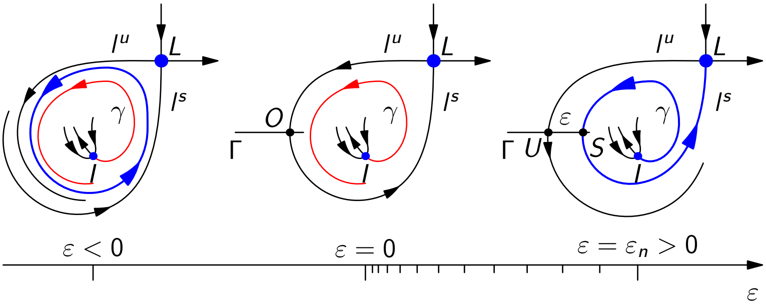
Universidade Federal do Rio de Janeiro



Asymptotics of ε_n

$$\lim_{n\to\infty} \frac{1}{n} \ln(-\ln \varepsilon_n) \to \ln \lambda,$$

where λ is the *characteristic number* of the saddle, i.e. the absolute value of the ratio of its eigenvalues, the negative one is in the numerator.
Reason: Poincaré map has asymptotics $P(x) \approx x^\lambda$.

Arnold's conjectures, 1985

For a **generic** smooth k-parameter family v_ε of vector fields on S^2,

1. X bifurcation actually happens in a neighborhood of its *support*: certain finite set of singular trajectories;
2. X on neighborhood of the support, bifurcation is equivalent to one of finitely many;
3. X locally, bifurcation diagram is topologically equivalent to one of finitely many;
4. X for each degenerate vector field, there exists a versal deformation;
5. X v_ε is structurally stable: C^1-close families experience same bifurcation as v_ε.

$v_\varepsilon \sim w_\delta$ if for some change of parameter h, for all ε, phase portraits of v_ε and $w_{h(\varepsilon)}$ are topologically equivalent.

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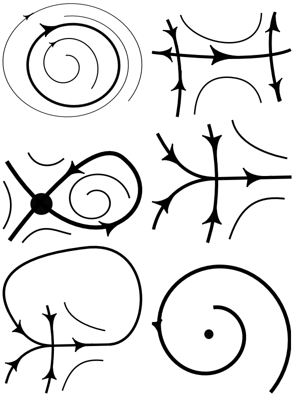

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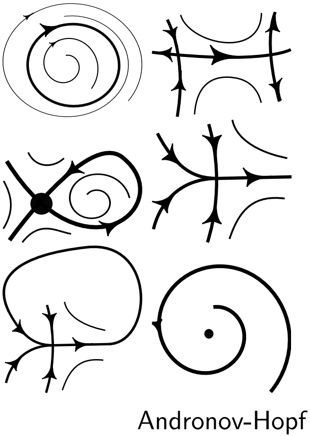

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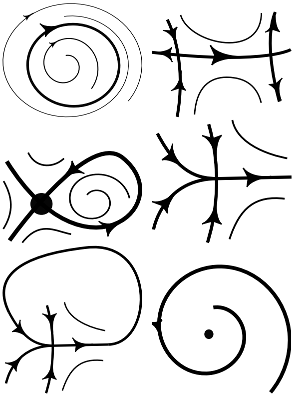

Andronov-Hopf

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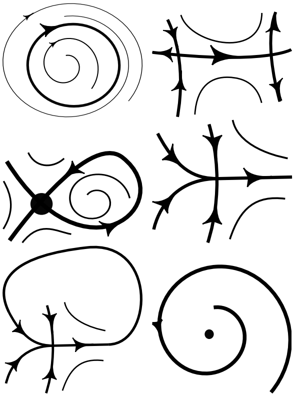

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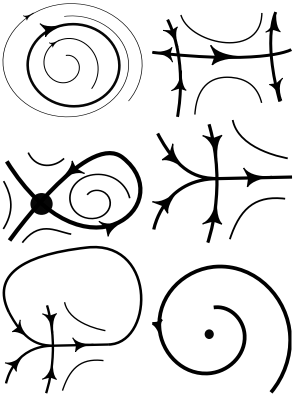

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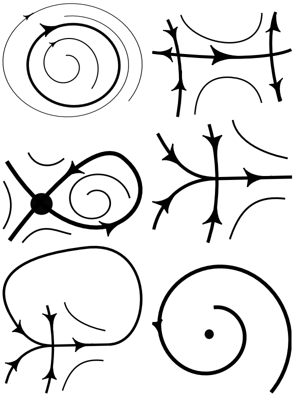

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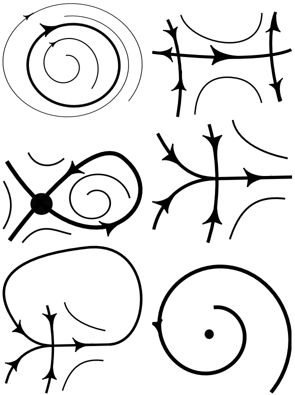

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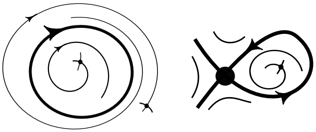

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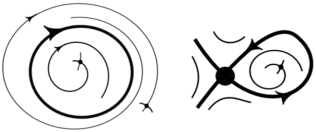

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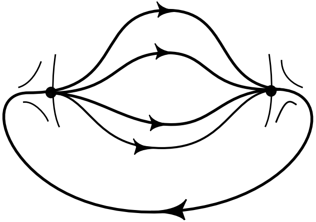

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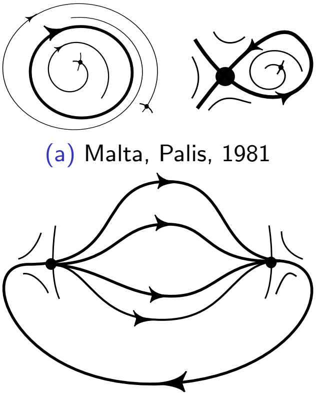

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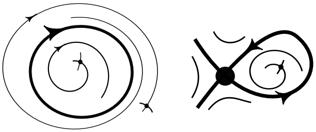

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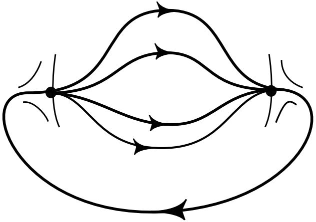

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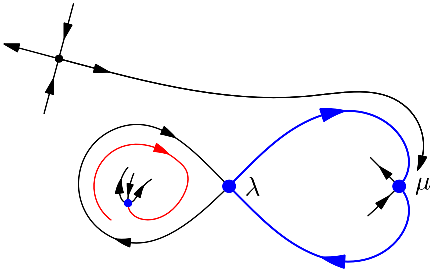
Structurally unstable family: Tear of the heart

Ilyashenko, Kudryashov, Schurov



- A degeneracy of codimension 3.
- Topological classification of unfoldings has at least one numerical invariant.

Theorem

Let v_0 and w_0 be two vector fields, each with a polycycle "tear of the heart". Let v_ε, w_δ be two 3-parameter unfoldings of v_0 and w_0, respectively. If v_ε is moderately equivalent to w_δ, then

$$\frac{\ln \lambda(v_0)}{\ln \mu(v_0)} = \frac{\ln \lambda(w_0)}{\ln \mu(w_0)}.$$

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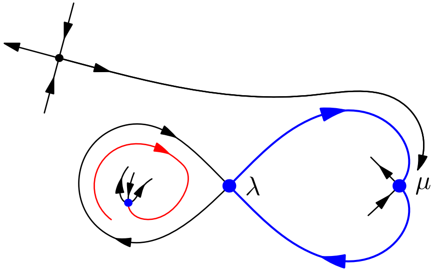

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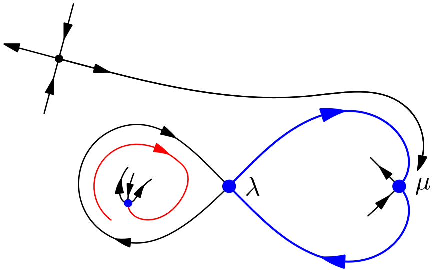

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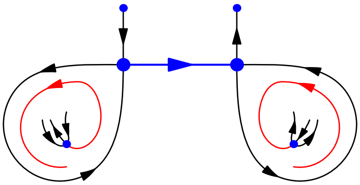
Structurally unstable families: "Glasses"



- Take a generic 3-parameter unfolding of "glasses".
- We can have separatrix connections on the left.
- Or on the right.
- And we have a "synchronizing" subfamily.

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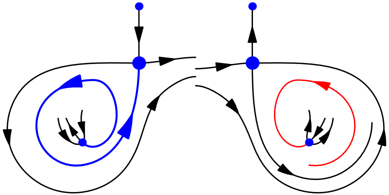

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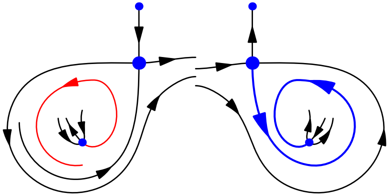

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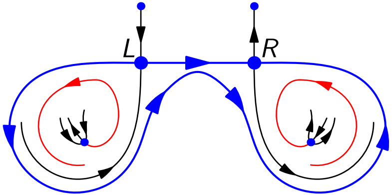

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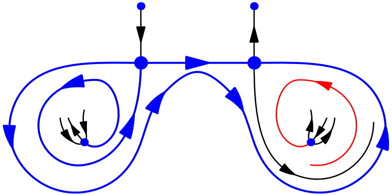

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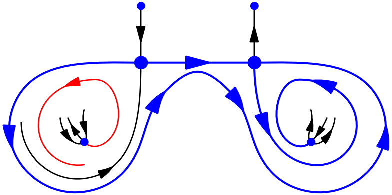

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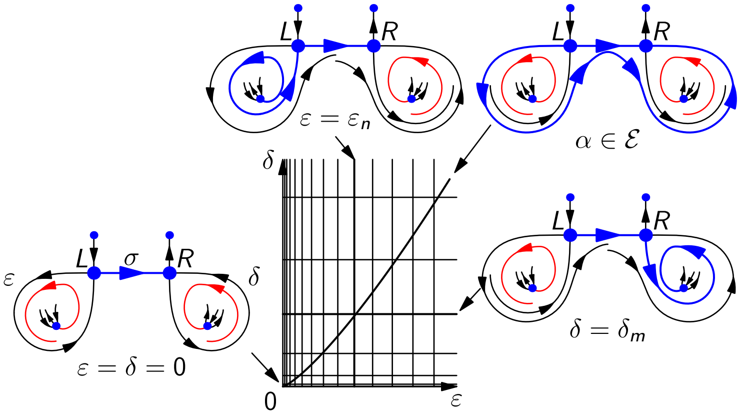
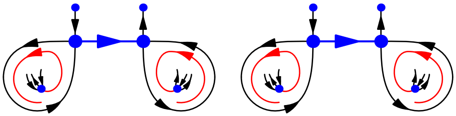
Partial bifurcation diagram of the "glasses"



$$\frac{m}{n} \approx \frac{\ln(-\ln \delta_m)}{|\ln \rho|} \div \frac{\ln(-\ln \varepsilon_n)}{\ln \lambda} = \frac{\ln(-\ln \delta_m)}{\ln(-\ln \varepsilon_n)} \frac{\ln \lambda}{|\ln \rho|} \approx \frac{\ln \lambda}{|\ln \rho|}$$

Functional invariants: two copies of "glasses"



- Take a vector field with 2 copies of "glasses".
- Take a 7-parameter unfolding.
- On a curve $\gamma \subset (\mathbb{R}^7, 0)$ both "glasses" survive.
- Image of this curve under (φ_1, φ_2) is a functional invariant.

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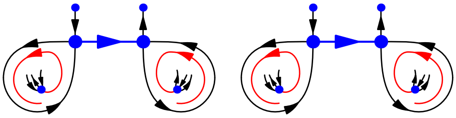

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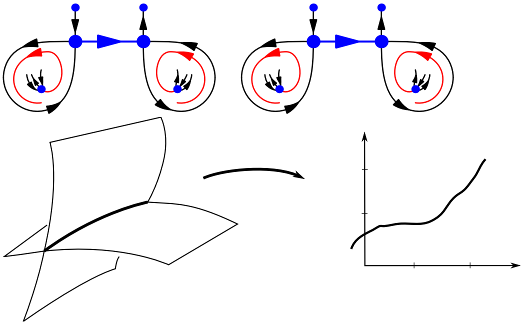

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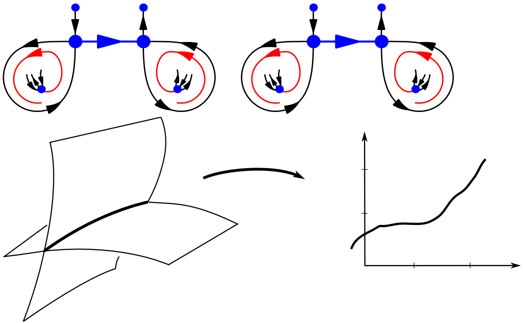

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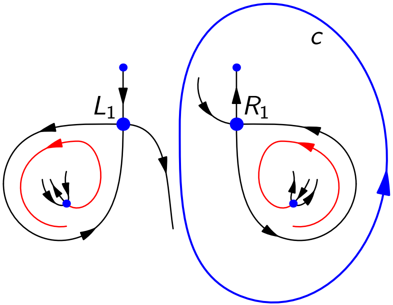
Whirled glasses: infinitely many invariants



- Replace separatrix connection with a parabolic cycle.
- If we destroy the parabolic cycle, we get a sequence of separatrix connections.
- Sequence of the values of φ at these vector fields is an invariant.
- The value $\varphi(v) = -\frac{\ln \lambda}{\ln \rho}$ for the unperturbed vector field is an invariant too.

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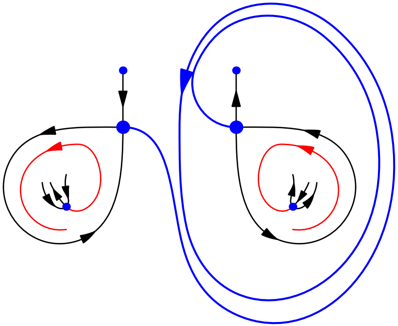

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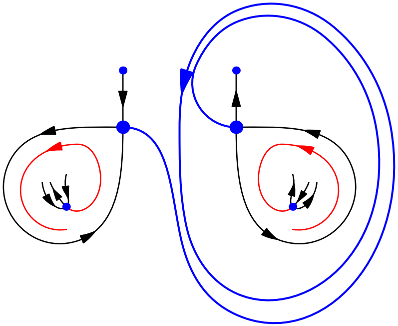

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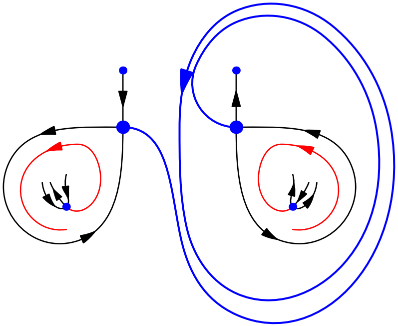

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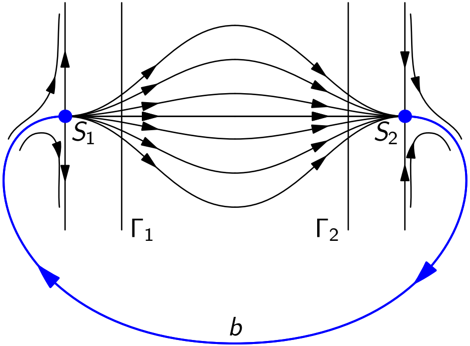

- Consider ensemble "lips" (codim $= 3$).
- Add $d + 1$ Cherry cells inside and one "glass lense" outside (now codim $= 4$).
- We can destroy "lips" so that one of d "glasses" appear.
- d numerical invariants.
- Functional invariants in unfoldings with $\leq d + 3$ parameters.

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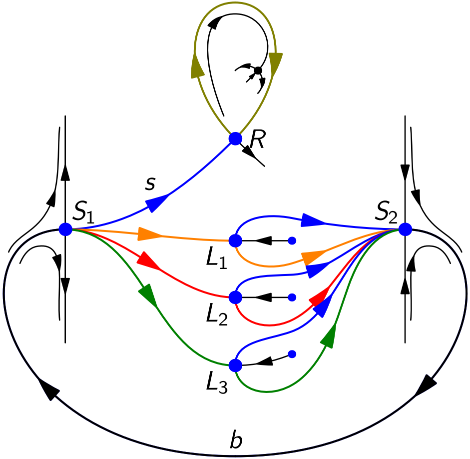

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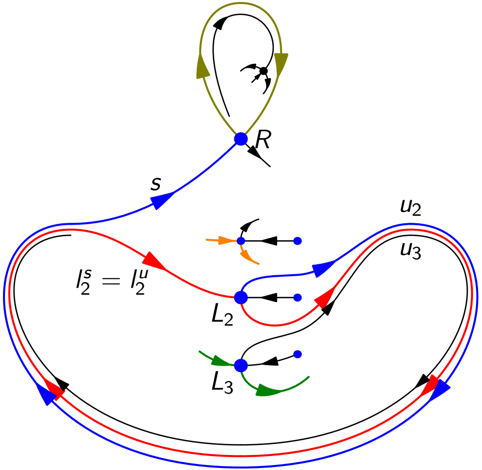

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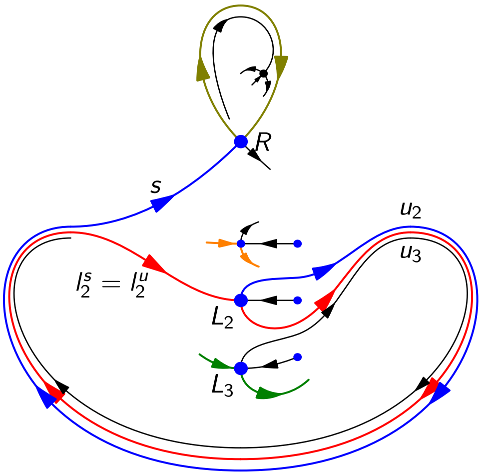
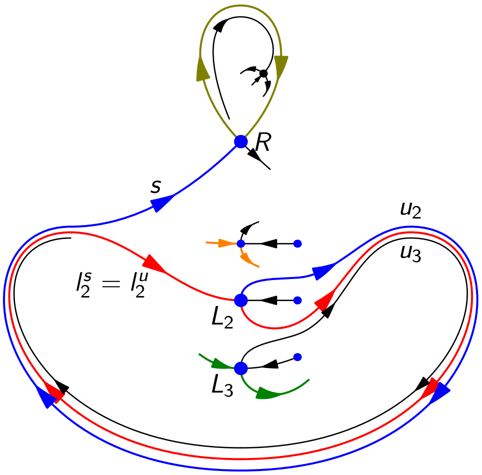


- Consider ensemble “lips” (codim $= 3$).
- Add $d + 1$ Cherry cells inside and one “glass lense” outside (now codim $= 4$).
- We can destroy “lips” so that one of d “glasses” appear.
- d numerical invariants.
- Functional invariants in unfoldings with $\leq d + 3$ parameters.

Lips with glasses



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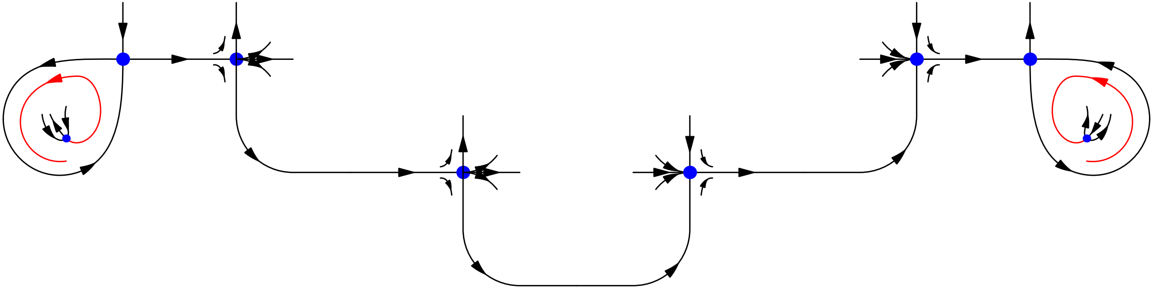
A family with no versal deformations



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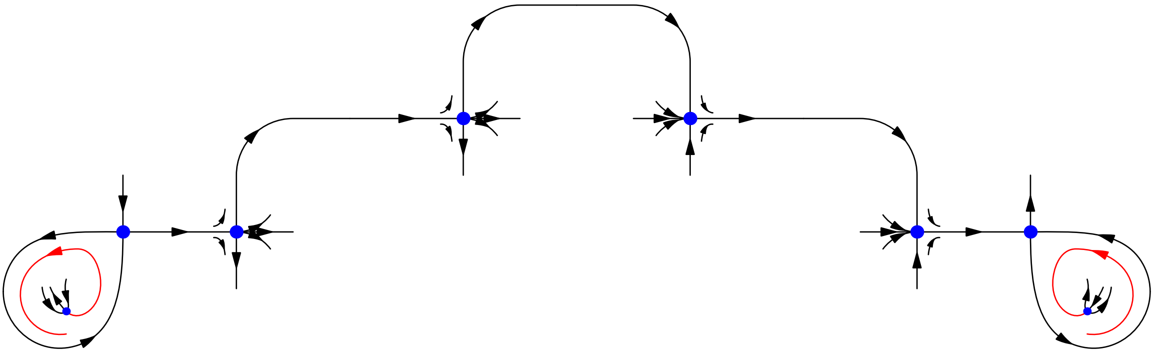
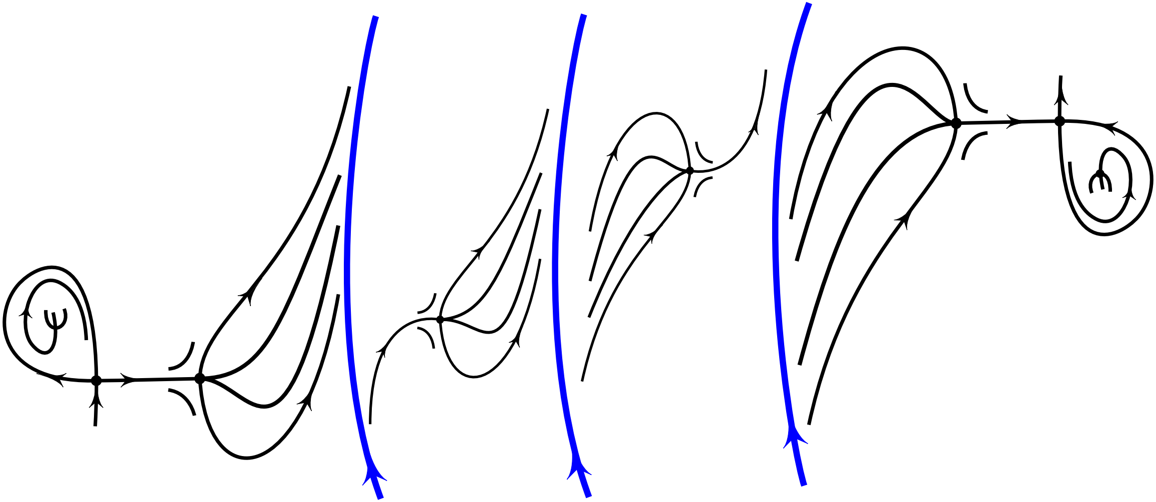


A family with no versal deformations



A family with no versal deformations