MATH 308. Differential Equations

Lecture 10: Second-order nonhomogeneous equations (Sec. 3.5, 3.6)

Nataliya Goncharuk

Texas A & M

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Structure of solutions

Theorem

If y_p is a particular solution of $y'' + p(t)y' + q(t)y = r(t)$ and y_c is the general solution of $\quad y'' + p(t)y' + q(t)y = 0$ **(complementary function)**, *then*

$$y = y_c + y_p$$

is the general solution of $y'' + p(t)y' + q(t)y = r(t)$.

Example

$y'' = y + e^{2t}$

- $y_p = ke^{2t}$ is a particular solution for $k = 1/3$ (substitute y to find k).
- $y_c = c_1 e^t + c_2 e^{-t}$ is the general solution of $y'' = y$.
- $y(t) = y_p + y_c = 1/3e^{2t} + c_1 e^t + c_2 e^{-t}$.

Method of undetermined coefficients

Guessing table for y_p

$$y'' + ay' + by = f(t)$$

Guessing table

$f(t)$	y_p
$ce^{\lambda t}$	$ke^{\lambda t}$
$c \sin at$	$k \sin at + l \cos at$
$c \cos at$	$k \sin at + l \cos at$
Polynomial $p(t)$	Polynomial $q(t)$, same degree
$p(t)e^{\lambda t} \sin at$	$q_1(t)e^{\lambda t} \sin at + q_2(t)e^{\lambda t} \cos at$
$p(t)e^{\lambda t} \cos at$	$q_1(t)e^{\lambda t} \sin at + q_2(t)e^{\lambda t} \cos at$

(Resonant case): If some terms of your guess appear in the complementary function, it will not work.

Rule: multiply your guess by t. If situation persists, multiply by t^2.

Examples

Example

$y'' + 3y' + 2y = e^t + \cos t$

Complementary function: $y_c(t) = c_1 e^{-t} + c_2 e^{-2t}$ What is a good guess of a particular solution?

(A) $k_1 e^t + k_2 \cos t$

(B) $k_1 e^t + k_2 \cos t + k_3 \sin t$

(C) $k_1 e^t + k_2 \sin t$

(D) IDK

https://pingo.coactum.de/885803

[Finding $y(t)$ on the whiteboard]

Example

$y'' + 3y' + 2y = e^{-t}$.

- The guess $y_p = ke^{-t}$ fails since it appears in y_c.
- **Multiply by t:** kte^{-t} works for some k.

Find k such that kte^{-t} is a particular solution.

(A) 1 (B) -1 (C) 2 (D) IDK

Variation of Parameters, $y'' + p(t)y' + q(t)y = r(t)$

- The general solution of the homogeneous equation is $y(t) = C_1 y_1(t) + C_2 y_2(t)$.
- **Variate constants,** i.e. replace C_1, C_2 with unknown functions: $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$.
- Add this condition to simplify computations:

$$u_1' y_1 + u_2' y_2 = 0 \quad (1)$$

  Then $y'(t) = u_1 y_1' + u_2 y_2'$.
- The equation $y'' + p(t)y' + q(t)y = r(t)$ is one more equation on u_1, u_1', u_2, u_2'. But u_1, u_2 cancel out. We get

$$u_1' y_1' + u_2' y_2' = r(t) \quad (2)$$

- We get two equations on u_1', u_2'. Now express $u_1', u_2' \to$ integrate to find $u_1, u_2 \to$ write out $y(t)$.

Example: $y'' - 2y' + y = t^{10}e^t$

- The general solution of the homogeneous equation is $y(t) = C_1 e^t + C_2 te^t$.
- Replace constants with unknown functions: $y(t) = u_1 e^t + u_2 te^t$.
- Add an extra equation: $u_1' e^t + u_2' te^t = 0 \qquad (1)$.
- Compute y', y'': $y'(t) = u_1 e^t + u_2(te^t)'$, $y''(t) = u_1' e^t + u_1 e^t + u_2'(te^t)' + u_2(te^t)''$.
- Substitute: $u_1' e^t + \cancel{u_1 e^t} + u_2'(te^t)' + \cancel{u_2(te^t)''} - \cancel{2 \cdot u_1 e^t} - \cancel{2 \cdot u_2(te^t)'} + \cancel{u_1 e^t} + \cancel{u_2 te^t} = t^{10}e^t \qquad (2)$.
- So we have

$$u_1' e^t + u_2' te^t = 0$$
$$u_1' e^t + u_2'(te^t + e^t) = t^{10}e^t.$$

- Solve for u_1', u_2': $u_1' = -t^{11}$, $u_2' = t^{10}$.
- Integrate: $u_1 = -t^{12}/12 + C_1$, $u_2 = t^{11}/11 + C_2$.
- Substitute: $\boxed{y(t) = C_1 e^t + C_2 te^t - \frac{t^{12}}{12} e^t + \frac{t^{11}}{11} \cdot te^t}$.