MATH 308. Differential Equations

Lecture 11: Forced Oscillators (Sec. 3.8)

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Harmonic oscillator: $my'' = -ky$

Amplitude and phase shift

$$y'' + 4y = 0$$

[Plotting graphs of solutions $y(t) = A\cos(2t) + B\sin(2t)$]

- All graphs look like a cosine wave.
- Find R and δ such that

$$A\cos(2t) + B\sin(2t) = R\cos(2t - \delta).$$

- $A\cos(2t) + B\sin(2t) = R\cos\delta\cos(2t) + R\sin\delta\sin(2t)$.
- $R = \sqrt{A^2 + B^2}$, $\tan\delta = \frac{B}{A}$.
- R is called the *amplitude*, δ is the *phase angle*.

Example

Find the amplitude of the solution $3\cos(2t) + 4\sin(2t)$.

(A) 5 (B) 7 (C) 25 (D) IDK

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Forced harmonic oscillator: $my'' = -ky + F_{out}(t)$

$$y'' + y = \sin \omega t.$$

Example

Find the particular solution using the Undetermined Coefficients method.

(A) $\frac{1}{\omega} \sin \omega t$;

(B) $\frac{1}{1-\omega} \cos \omega t$;

(C) $\frac{1}{1-\omega^2} \sin \omega t$;

(D) $\frac{1}{1-\omega^2} \sin \omega t + \frac{1}{1-\omega^2} \cos \omega t$.

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[Demonstrating
https://www.desmos.com/calculator/qijbhbcefc — graphs of solutions with fixed initial conditions for different ω]

- For $\omega = \pm 1$, we have resonance: frequency of F_{out} coincides with the natural frequency of the system, and the amplitude of the solution $1/(1-\omega^2)$ tends to infinity.

Forced harmonic oscillator

Solution in the resonant case

$$y'' + y = \sin t.$$

Example

What is the correct guess for the particular solution?

(A) $a\cos t$;

(B) $a\cos t + b\sin t$;

(C) $at\cos t$;

(D) $at\cos t + bt\sin t$;

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[plotting solutions with fixed initial conditions]

Damped oscillator [reminder]

$$my'' = -ky - cy', \qquad c > 0$$

Characteristic equation: $\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$; let $D = \frac{c^2}{m^2} - 4\frac{k}{m}$ be the discriminant.

Underdamped oscillator – complex roots $-\frac{1}{2}\frac{c}{m} \pm i\frac{1}{2}\sqrt{|D|}$;

Solution: $y(t) = Ae^{-\frac{1}{2}\frac{c}{m}\cdot t}\cos(\sqrt{D}t - \delta)$;

Critically damped oscillator – 1 negative real root $\lambda = -\frac{1}{2}\frac{c}{m}$;

Solution: $y(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$

Overdamped oscillator – 2 negative real roots $\lambda_{1,2} < 0$.

Solution: $y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

Example

When the limit of $y(t)$ as $t \to +\infty$ is zero?

(A) For the overdamped case. (B) For the underdamped case.

(C) For both. (D) IDK.

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Forced damped oscillator

$$my'' + cy' + ky = F_0 \cos(\omega t)$$

General solution: $y(t) = y_c(t) + y_p(t)$, where

- y_c is *transient solution*. **It tends to zero** as $t \to +\infty$ as in the previous slide.
- y_p is *forced responce*:
  $y_p(t) = A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$ for some A, B.
  It is periodic.
- **Conclusion:** y approaches a periodic function y_p as $t \to +\infty$.
- **Remark:** Terms from y_p never appear in y_c, so the guess always works.
- **Resonance:** Amplitude of y_p depends on ω and may abruptly increase for some ω. You are asked to find the "most resonant" ω in HW5, Task 2.