MATH 308. Differential Equations

Lecture 12: Laplace transform (Sec. 6.1, 6.2)

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Discontinuous ODEs and impacts

The model for forced oscillators and electrical circuits:

$$y'' + ay' + by = f(t)$$

- **Switching the circuit on/off:** discontinuous $f(t)$.
- **Impact for mechanical oscillators:** $f(t)$ is infinity at some point, but has a finite integral.

Model for impacts

$$y'' + ay' + by = k\delta(t)$$

The δ-function is a generalized function that is ∞ at zero, 0 elsewhere, and has integral equal to 1.

To deal with these, we introduce **Laplace transforms.**

Laplace transform: phylosophy

- Laplace transform turns functions into other functions:

$$f(t) \to \mathcal{L}\{f\} = F(s).$$

- e.g. $\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{\cos t\} = \frac{s}{1+s^2}, \quad \mathcal{L}\{\delta(t)\} = 1.$
- $\mathcal{L}$ transforms differentiation into multiplication by s:

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0).$$

- Differential equations turn into algebraic equations that can be solved.
- Inverse Laplace transform $\mathcal{L}^{-1}$ transforms functions back: $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1, \quad \mathcal{L}^{-1}\{\frac{s}{1+s^2}\} = \cos t, \ldots$
- The formula for $\mathcal{L}^{-1}$ is not in this course.
- We will use the Table of Laplace transforms (p.252)

Laplace transform: definition

Definition

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t) dt$$

Comment: $\mathcal{L}\{f\}$ is a function of s. It is defined for such s that the improper integral converges.

Laplace transform: examples

Definition

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)dt$$

Example

Compute $\mathcal{L}\{1\}$:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_{t=0}^{t=\infty} = \frac{1}{s}$$

if $s > 0$ (otherwise undefined)

Example

Compute $\mathcal{L}\{e^{2t}\}$.

(A) $\frac{e^{2t}}{s}$; (B) $\frac{1}{s-2}$; (C) $\frac{1}{2s}$; (D) $\frac{1}{2} + \frac{1}{s}$.

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Laplace transform: domain of definition

Definition

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t) dt$$

Functions of at most exponential growth

If $|f(t)| < ce^{at}$, then $\mathcal{L}\{f\}$ is defined for $s > a$.

Proof

If $|f(t)| < ce^{at}$, then $|f(t)e^{-st}| < |ce^{(a-s)t}|$.
The improper integral of $e^{(a-s)t}$ converges if $s > a$.
Thus the integral of $|f(t)e^{-st}|$ also converges (Comparison test).

Example

The Laplace transform of $e^t \sin t$ is guaranteed to be defined for:

(A) $s > -1$; (B) $s > 0$; (C) $s > 1$; (D) $s > 2$.

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Laplace transform: linearity

$$\mathcal{L}\{f+g\} = \int_0^\infty e^{-st}(f(t)+g(t))dt = \mathcal{L}\{f\} + \mathcal{L}\{g\}$$

Example

$\mathcal{L}\{f+2\}$ is equal to:

(A) $\mathcal{L}\{f\}+2$;

(B) $\mathcal{L}\{f\}+\frac{2}{s}$;

(C) $\mathcal{L}\{f\}+2e^{-st}$;

(D) IDK.

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You can also multiply by a constant:

$$\mathcal{L}\{cf\} = \int_0^\infty e^{-st}\cdot cf(t)dt = c\cdot\mathcal{L}\{f\}$$

Laplace transform and differentiation

Theorem

If $f(t) < c|e^{at}|$, then for $s > a$, $\boxed{\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)}$

Proof.

Integrating by parts:

$$\mathcal{L}\{f'\} = \int_0^\infty e^{-st} f'(t)dt =$$

$$e^{-st} f(t)\Big|_0^{+\infty} + \int_0^\infty s e^{-st} f(t)dt = -f(0) + s\mathcal{L}\{f\}$$

The term $e^{-s \cdot t} f(t)\Big|^{+\infty}$ is zero for $s > a$, since $f(t) < c|e^{at}|$. □

Laplace transform and differentiation: example

Theorem

If $f(t) < c|e^{at}|$, then for $s > a$, $\boxed{\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)}$

Example

The Laplace transform of $\cos t$ is $\dfrac{s}{1+s^2}$. Find the Laplace transform of $\sin t$.

(A) $\dfrac{s^2}{1+s^2}$ (B) $\dfrac{s^2}{1+s^2} - 1$ (C) $1 - \dfrac{s^2}{1+s^2}$ (D) $-\dfrac{s^2}{1+s^2}$

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Laplace transform and solutions of the ODEs

$$y'' + y = 0, \qquad y(0) = 1, y'(0) = 0$$

- Take Laplace transforms of both sides:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 0$$

- $\mathcal{L}\{y''\} = s\mathcal{L}\{y'\} - y'(0) =$
- $= s^2\mathcal{L}\{y\} - sy(0) - y'(0)$, so
- $$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = 0$$
- $$(s^2 + 1)\mathcal{L}\{y\} = sy(0) + y'(0) = s$$
- **Conclusion.** Laplace transform of $\cos t$ equals $\dfrac{s}{s^2 + 1}$.