MATH 308. Differential Equations

Lecture 13: Laplace transform (Sec. 6.2, 6.3)

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Laplace transform: reminder

Definition

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t) dt$$

$\mathcal{L}\{1\} = \frac{1}{s}, \mathcal{L}\{e^{at}\} = \frac{1}{s-a},$
$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}, \mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$

- Linearity: $\boxed{\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}}$
- Differentiation becomes $\cdot s$: $\boxed{\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)}$.

Using the Table to invert Laplace transforms

Example

Compute the inverse Laplace transform of $\dfrac{s}{s^2+4s+8}$

- Complete the square: $\dfrac{s}{s^2+4s+8} = \dfrac{s}{(s+2)^2+4}$.
- Represent as a sum of functions in the table: $\dfrac{s+2}{(s+2)^2+4} - \dfrac{2}{(s+2)^2+4}$.
- Answer: $\mathcal{L}^{-1}\{\frac{s}{s^2+4s+8}\} = e^{2t}\cos t - e^{2t}\sin t$.

Example

Compute the inverse Laplace transform of $\dfrac{1}{s^2+6s+9}$.

(A) e^{3t}; (B) $0.5t^2e^{3t}$; (C) $e^{3t}\sin t$; (D) $e^{3t}\sin t - e^{3t}\cos t$.

https://pingo.coactum.de/885803

Using the Table to invert Laplace transforms

Example

Compute the inverse Laplace transform of $\dfrac{s+1}{s^2-3s+2}$.

- Use partial fractions: $\dfrac{s+1}{s^2-3s+2} = \dfrac{a}{s-1} + \dfrac{b}{s-2}$.
- $a(s-2) + b(s-1) = s+1 \Rightarrow a+b=1, -2a-b=1$
- $a = \frac{1}{3}, b = \frac{2}{3}$
- Answer: $\frac{1}{3}e^{t} + \frac{2}{3}e^{2t}$.

Step functions

Definition

The Heaviside function is

$$u_c(t) = \begin{cases} 0, t < c \\ 1, t > c \end{cases}$$

[plotting a graph]

We will only consider $c > 0$.

Example

Plot the graph of $1 - u_1(t)$.

Example

Plot the graph of $u_1(t) - u_2(t)$.

Example

Represent a function [plot the graph] as a sum of Heaviside functions and constants.

Step functions

Definition

The Heaviside function is

$$u_c(t) = \begin{cases} 0, t < c \\ 1, t > c \end{cases}$$

Example

Write down the formula for a function [plot the graph] using Heaviside functions and constants.

Shifting via Heaviside functions and Laplace transforms

- **Shifted function:** $u_c(t)f(t-c)$, $c > 0$.
- Plotting the graph of the shifted function $u_c(t)f(t-c)$, $c > 0$.

Theorem

$\boxed{\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f\}}$ *for* $c > 0$.

Proof.

$\mathcal{L}\{u_c(t)f(t-c)\} = \int_c^\infty e^{-st}f(t-c)dt = \int_c^\infty e^{-s(t-c)-sc}f(t-c)d(t-c) = \int_0^\infty e^{-s\tau-sc}f(\tau)d\tau = e^{-cs}\mathcal{L}\{f\}$. □

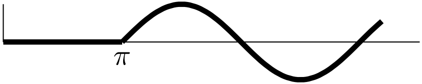
Shifting via Heaviside functions and inverse Laplace transforms

$$\boxed{\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f\}}$$

Example

Compute the inverse Laplace transform of $\frac{e^{-\pi s}}{s^2+1}$.

Answer: shifted sine,



Example

Compute the inverse Laplace transform of $\dfrac{e^{-2s}}{s^2}$.

[plotting the graph]

ODEs with discontinuous right-hand sides

$$y'' = u_5(t), \quad y(0) = 0, y'(0) = 1$$

- Apply Laplace transform to both sides:

$$s^2\mathcal{L}\{y\} - sy'(0) - y(0) = \frac{e^{-5s}}{s}$$

- Substitute initial conditions, find $\mathcal{L}\{y\}$:

$$\mathcal{L}\{y\} = \frac{e^{-5s}}{s^3} + \frac{1}{s}$$

- Compute $\mathcal{L}^{-1}$:
- $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$, $\mathcal{L}^{-1}\{\frac{1}{s^3}\} = 0.5t^2$, so

$$y(t) = 1 + u(t-5)\left(0.5(t-5)^2\right)$$

- Plot the graph and enjoy!