MATH 308. Differential Equations

Lecture 14: Laplace transform (Sec. 6.4)

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Laplace transforms and derivatives

- Differentiation becomes $\cdot s$: $\boxed{\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)}$.

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$$(\mathcal{L}\{f\})' = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt =$$

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Example

Compute $\mathcal{L}\{t^2e^{2t}\}$ knowing that $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$.

(A) $\frac{2}{(s-2)^3}$; (B) $\frac{1}{(s-2)^2}$; (C) $\ln|s-2|$;

(D) $(s-2)\ln|s-2| - (s-2)$.

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$$\mathcal{L}\{e^{ct}f(t)\} = \int_0^{\infty} e^{ct}e^{-st}f(t)dt = \qquad .$$

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Example

Compute $\mathcal{L}\{e^t \sin t\}$ knowing that $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$.

(A) $\frac{1}{s^2}$; (B) $\frac{1}{s^2+1} - 1$; (C) $\frac{1}{(s-1)^2+1}$; (D) IDK.

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ODEs with discontinuous right-hand sides

$$y'' = u_5(t), \quad y(0) = 1, y'(0) = 0$$

- Apply Laplace transform to both sides:

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) = \frac{e^{-5s}}{s}$$

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- Plot the graph and enjoy!

Example

$$y' + y = 2u_1(t), \quad y(0) = 1$$

Example

Apply Laplace transform to both sides.

(A) $s^2\mathcal{L}\{y\} - 1 + \mathcal{L}\{y\} = 2\frac{e^{-s}}{s}$

(B) $s\mathcal{L}\{y\} - 1 + \mathcal{L}\{y\} = 2\frac{e^{-s}}{s}$

(C) $s\mathcal{L}\{y\} - 1 + \mathcal{L}\{y\} = \frac{e^{-2s}}{s}$

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Also, express $\mathcal{L}\{y\}$. https://pingo.coactum.de/885803

Example: $y' + y = 2u_1(t), \quad y(0) = 1$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2e^{-s}}{s(1+s)} + \frac{1}{s+1}\right\}.$$

Example

Find $\mathcal{L}^{-1}\{\frac{2}{s(s+1)}\}$ using partial fractions.

(A) $2t(1-t)$;

(B) $2 + 2e^{t}$;

(C) $2 - 2e^{-t}$;

(D) $2e^{-t}$.

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Example

Now find $y(t)$.

(A) $2 - 2e^{-t+1} + e^{-t}$;

(B) $2u_1(t) \cdot (1 - e^{-t+1}) + e^{-t}$;

(C) $2u_1(t) \cdot (1 - e^{-t-1}) + e^{-t}$;

(D) $2u_1(t-1) \cdot (1 - e^{-t+1}) + e^{-t}$;

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(C) $2u_1(t) \cdot (1 - e^{-t-1}) + e^{-t}$;

(D) $2u_1(t-1) \cdot (1 - e^{-t+1}) + e^{-t}$;

Is your answer continuous?

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General Case

$y'' + ay' + by = f(t)$
Suppose that $f(t)$ only has jump disontinuities.

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The answer $y(t)$ will:

- Satisfy the equation whenever f is continuous;
- y, y' will be continuous;
- y'' will have a jump discontinuity when f has a jump discontinuity.

Example-2

$$y'' + y = 1 - u_{3\pi}(t), \quad y(0) = y'(0) = 0.$$

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Example

Find $\mathcal{L}^{-1}\{\frac{1}{s(1+s^2)}\}$ (use partial fractions).

(A) $t - \cos t$; (B) $1 - \cos t$; (C) $1 + \cos t$; (D) $t \cos t$.

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Example

Find $\mathcal{L}^{-1}\{\frac{e^{-3\pi s}}{s(1+s^2)}\}$.

(A) $1 - \cos(t - 3\pi)$;

(B) $u_3(t)(1 - \cos(t - 3\pi))$;

(C) $1 + u_3(t)\cos(t - 3\pi)$;

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Also write out $y(t) = \mathcal{L}^{-1}\{\frac{1-e^{-3\pi s}}{s(1+s^2)}\}$.

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[Plotting the graph of y, making sure it satisfies the equation and y, y' are continuous]