MATH 308. Differential Equations

Lecture 15: Laplace transform (Sec. 6.4, 6.5)

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- Plotting the graph of $y(t)$ using desmos.com (use {} to specify domains of the functions).
- Interpretation: y is the temperature in the house, $2u_1(t)$ is the outdoor temperature.

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- y, y' (derivatives of smaller order) will be continuous.

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Example

Find $\mathcal{L}^{-1}\{\frac{1}{s(1+s^2)}\}$ (use partial fractions).

(A) $t - \cos t$; (B) $1 - \cos t$; (C) $1 + \cos t$; (D) $t \cos t$.

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(A) $1 - \cos(t - 3\pi)$;

(B) $u_{3\pi}(t)(1 - \cos(t - 3\pi))$;

(C) $1 + u_{3\pi}(t)\cos(t - 3\pi)$;

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[Plotting the graph of y, making sure it satisfies the equation and y, y' are continuous]

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- $f_\varepsilon(t) = \dfrac{A}{2\varepsilon} u_{c-\varepsilon}(t) - \dfrac{A}{2\varepsilon} u_{c+\varepsilon}(t)$

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- Answer: $\boxed{\mathcal{L}\{\delta(t-c)\} = e^{-cs}.}$

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Example

$y'' + y = 3\delta(t-1), y(0) = 1, y'(0) = 0$

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- Plotting the graph of y: the function y is continuous at 1, but y' is not.
- The RHS of the equation can contain several δ-functions and Heaviside functions, the strategy is the same.

Example: antiderivative of the delta-function

Example

Solve: $y' = \delta(t-1)$, $y(0) = 0$.

(A) $y(t) = \frac{e^{-t}}{t}$;

(B) $y(t) = \delta(t-1)$;

(C) $y(t) = u_1(t)$;

(D) $y(t) = e^{-t}$.

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