Lecture 17: Power series (Sec. 5)

MATH 308. Differential Equations

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Power series and their sums

Example

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

This only holds for $|x| < 1$.

Example

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

This holds for all x.
[Substituting $x = 0$ and $x = 1$]

Power series centered at zero: general case

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n = f(x)$$

- a_n are **coefficients** of the power series.
- $f(x)$ is the **sum** of power series.
- The series converges to $f(x)$ only for $|x| < R$ where $R \in [0, +\infty]$ is a **radius of convergence** of power series. For $|x| > R$, the series diverges.
- $|x| < R$ is the **interval of convergence.**
- $R = 0$ means that the series only converges for $x = 0$.
- $R = \infty$ means that the series converges for all x.

Power series centered at x_0: general case

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n = f(x)$$

- a_n are **coefficients** of the power series.
- $f(x)$ is the **sum** of power series.
- The series converges to $f(x)$ only for $|x - x_0| < R$ where $R \in [0, +\infty]$ is a radius of convergence of power series. For $|x - x_0| > R$, the series diverges.
- $|x - x_0| < R$ is the **interval of convergence.**
- $R = 0$ means that the series only converges for $x = x_0$.
- $R = \infty$ means that the series converges for all x.

Remark

Power series centered at the same point are equal if and only if all coefficients are equal.

Taylor series as power series

Many functions can be represented as a sum of power series called **Taylor series**:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \dots$$

Coefficients of the Taylor series are $a_n = \dfrac{f^n(x_0)}{n!}$.

- $e^x = 1 + x + \frac{x^2}{2!} + \dots$, for all x ($R = \infty$)
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$, for all x ($R = \infty$)
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$, for all x ($R = \infty$)
- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$, for $|x| < 1$ ($R = 1$).
- $\frac{1}{1-x} = 1 + x + x^2 + \dots$, for $|x| < 1$ ($R = 1$).

Power series coincides with the Taylor series of its sum.

Operations with power series: algebraic operations

Only work inside the interval of convergence

- Add: $\frac{2}{1-x} + e^x = \sum_0^\infty (2 + \frac{1}{n!})x^n$ for $|x| < 1$
- Multiply:
  $\frac{1}{(1-x)^2} = (1 + x + x^2 + \dots)(1 + x + x^2 + \dots) =$
  $1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=0}^\infty (n+1)x^n.$
  for $|x| < 1$
- Substitute: $\frac{1}{1-2x^2} = 1 + (2x^2) + (2x^2)^2 + \dots =$
  $1 + 2x^2 + 4x^4 + 8x^6 + \dots = \sum_{n=0}^\infty 2^n x^{2n}.$
  for $|2x^2| < 1$

Example

Compute several first terms of the Taylor series for $1 + \frac{x^3}{1-x^2}$.

(A) $x^3 + x^8 + x^{13} + \dots$

(B) $1 + x^3 - x^5 + x^7 - \dots$

(C) $1 + x^3 + x^5 + x^7 + \dots$

(D) $2 + x^6 + x^{12} + x^{18} + \dots$

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Operations with power series: analytic

- Differentiate:
  $(\frac{1}{(1-x)^2})' = \frac{2}{(1-x)^3} = (1 + 2x + 3x^2 + 4x^3 + \dots)'$
  thus
  $\frac{2}{(1-x)^3} = 2 \cdot 1 + 3 \cdot 2x + 4 \cdot 3x^2 + \dots = \sum_{n=0}^{\infty}(n+2)(n+1)x^n$.
- Integrate: $\ln(1+x) = \int \frac{1}{1+x} dx =$
  $\int(1 - x + x^2 - x^3 + \dots)dx = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$
  We put $C = 0$ since for $x = 0$, we must have $\ln(1+x) = 0$.

Example

Compute several first terms of the Taylor series for $\int e^{(x^2)} dx$.

(A) $x + \frac{x^3}{3} + \frac{x^5}{2! \cdot 5} + \frac{x^7}{3! \cdot 7} + \dots$

(B) $1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$

(C) $2x + 4\frac{x^3}{2!} + 6\frac{x^5}{3!} + \dots$

(D) $x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 3!} + \dots$

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Solving differential equations with power series

Find the solution of the equation $y'' = xy, y(0) = 1, y'(0) = 0$ in the form $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$.

- Substitute into the equation:

$$2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + \dots = a_0x + a_1x^2 + a_2x^3 + \dots$$

- Equate coefficients:

$$2a_2 = 0, \quad 2\cdot 3a_3 = a_0, \quad 3\cdot 4a_4 = a_1, \quad 4\cdot 5a_5 = a_2 \dots$$

- Use initial conditions: $a_0 = y(0) = 1$, $a_1 = y'(0) = 0$.
- Find $a_2, a_3, \dots$:

$$a_2 = 0, \quad a_3 = \frac{a_0}{2\cdot 3} = \frac{1}{2\cdot 3}, \quad a_4 = \frac{a_1}{3\cdot 4} = 0,$$

$$a_5 = \frac{a_3}{4\cdot 5} = \frac{1}{2\cdot 3\cdot 4\cdot 5}, a_6 = \frac{a_4}{5\cdot 6} = 0, \dots$$

- $\boxed{y(x) = 1 + \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}x^5 + \dots}$.