Lecture 18: Power series (Sec. 5)

MATH 308. Differential Equations

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Power series method for solving ODEs

Find the solution of the equation $y'' = xy, y(0) = 1, y'(0) = 0$ in the form $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$.

- Substitute into the equation:

$$2a_2+3\cdot 2a_3x+4\cdot 3a_4x^2+5\cdot 4a_5x^3+\dots = a_0x+a_1x^2+a_2x^3+\dots$$

- Equate coefficients:

$$2a_2 = 0, \quad 2\cdot 3a_3 = a_0, \quad 3\cdot 4a_4 = a_1, \quad 4\cdot 5a_5 = a_2 \dots$$

- Use initial conditions: $a_0 = y(0) = 1$, $a_1 = y'(0) = 0$.
- Find $a_2, a_3, \dots$:

$$a_2 = 0, \quad a_3 = \frac{a_0}{2\cdot 3} = \frac{1}{2\cdot 3}, \quad a_4 = \frac{a_1}{3\cdot 4} = 0,$$

$$a_5 = \frac{a_2}{4\cdot 5} = 0, a_6 = \frac{a_3}{5\cdot 6} = \frac{1}{2\cdot 3\cdot 5\cdot 6}, \dots$$

- $\boxed{y(x) = 1 + \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 + \dots}$.

Example

Find the first 5 coefficients of the power series at 0 of the solution of the equation

$$y'' = (1+x)y, y(0) = 1, y'(0) = 0.$$

Example (Right-hand side)

Given $y(x) = a_0 + a_1x + a_2x^2 + \dots$, compute $(1+x)y$.

(A) $a_0 + x + a_1x + a_1 + a_2x^2 + a_2 + \dots$

(B) $a_0 + (a_0 + a_1)x + (a_1 + a_2)x^2 + (a_2 + a_3)x^3 + \dots$

(C) $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

(D) $a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \dots$

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Also, equate $y'' = (1+x)y$: write out that the coefficients at the same powers of x are equal.

Recurrent relations

Since $y(0) = 1, y'(0) = 0$, we have $a_0 = 1, a_1 = 0$ and

$$2a_2 = a_0, \quad 3 \cdot 2a_3 = a_0 + a_1, \quad 4 \cdot 3a_4 = a_1 + a_2, \dots$$

Compute a_2, a_3, a_4 and write out $y(x)$.

(A) $y(x) = 1 + x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \dots$

(B) $y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$

(C) $y(x) = 1 + \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{24}x^3 + \dots$

(D) $y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4}x^3 + \frac{1}{18}x^4 + \dots$

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Fuchs' theorem

Definition

A function $y(x)$ is called **analytic** at x_0 if its Taylor series at x_0 converges to $y(x)$ in some interval $|x - x_0| < R$.

Theorem

If p, q, r are analytic at a point x_0, and their Taylor series converge to them for $|x - x_0| < R$, then any solution of the second order equation

$$y'' + p(x)y' + q(x)y = r(x)$$

is also analytic at x_0, and is given by the power series $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ that also converges for $|x - x_0| < R$.

Using the Fuchs' theorem

$$(1-2x)y'' = y$$

Estimate the radius of convergence of the power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

- $(1-2x)y'' = y \quad \to \quad y'' = \frac{1}{1-2x} y.$
- Power series for the coefficient $-q(x) = \frac{1}{1-2x} = 1 + 2x + (2x)^2 + \dots$ converges for $|x| < 1/2$.
- Power series for the coefficient $p(x) = 0$ converges for all x.
- Conclusion: $y(x) = \sum a_n x^n$ converges for $|x| < 1/2$.

Example

$(1+x^2)y'' - y' + xy = 0$

Estimate the radius of convergence of the power series solution $y(x) = \sum a_n x^n$.

(A) at least 0;

(B) at least 1;

(C) at least 2;

(D) ∞.

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$\sum$ notation

- $(\sum_{n=0}^{\infty} a_n x^n)' = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1}$.
- $(\sum_{n=0}^{\infty} a_n x^n)'' = \sum_{n=2}^{\infty} a_n \cdot n(n-1) x^{n-2}$.
- Use this notation for $y'' = xy$:
- $\sum_{n=2}^{\infty} a_n \cdot n(n-1) x^{n-2} = \sum_{n=0}^{\infty} a_n x^{n+1}$.
- How do we equate coefficients? Index shift by 3: replace n by $n+3$ in the first sum.
- Now we have $\sum_{n=-1}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} = \sum_{n=0}^{\infty} a_n x^{n+1}$
- So $a_2 = 0$, and $(n+3)(n+2) a_{n+3} = a_n$ — general form of the recurrent relation.

Index shift

Example

Shift the index in $\sum_{n=0}^{\infty} 2^n a_n x^n$ so that we have x^{n+1} instead of x^n inside the sum.

Solution: formally replace all n by $n+1$:

$$\sum_{n=0}^{\infty} 2^n a_n x^n = \sum_{n+1=0}^{\infty} 2^{n+1} a_{n+1} x^{n+1} = 2^0 a_0 x^0 + \sum_{n=0}^{\infty} 2^{n+1} a_{n+1} x^{n+1}$$

Example

Shift the index in $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ so that we have x^n inside the sum.

(A) $2a_2 + \sum_{n=2}^{\infty} n(n-1)a_{n+2}x^n$ (B) $\sum_{n=0}^{\infty}(n-2)(n-3)a_{n-2}x^n$

(C) $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$ (D) $\sum_{n=0}^{\infty}(n+2)(n+1)a_n x^n$

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Application

Find the power series solution of $y'' - xy' - 2y = 0$, $y(0) = 1, y'(0) = 0$.

- $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0$
- Shift indices:
  $\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+2)a_n x^n = 0$
- Recurrent relation: $(n+1)a_{n+2} = a_n$
- $a_0 = 1, a_1 = 0$ from initial conditions.
- $0 = a_1 = a_3 = a_5 = \dots$ — odd coefficient are zero.
- $a_0 = 1, a_2 = 1, a_4 = \frac{1}{3}, a_6 = \frac{1}{3\cdot 5}, a_8 = \frac{1}{3\cdot 5\cdot 7}, \dots$
- $$y(x) = 1 + x^2 + \frac{1}{3}x^4 + \frac{1}{3\cdot 5}x^6 + \frac{1}{3\cdot 5\cdot 7}x^8 + \dots$$