Lecture 19: Power series (Sec. 5.3), Linear Systems of ODEs

MATH 308. Differential Equations

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November 02, 2022

Application of index shift

Find the power series solution of $y'' - xy' - 2y = 0,\ y(0) = 1, y'(0) = 0.$

- $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0$
- Shift indices:
  $\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+2)a_n x^n = 0$
- Recurrent relation: $(n+1)a_{n+2} = a_n$
- $a_0 = 1, a_1 = 0$ from initial conditions.
- $0 = a_1 = a_3 = a_5 = \ldots$ — odd coefficient are zero.
- $a_0 = 1, a_2 = 1, a_4 = \frac{1}{3}, a_6 = \frac{1}{3\cdot 5}, a_8 = \frac{1}{3\cdot 5\cdot 7}, \ldots$
- $$y(x) = 1 + x^2 + \frac{1}{3}x^4 + \frac{1}{3\cdot 5}x^6 + \frac{1}{3\cdot 5\cdot 7}x^8 + \ldots$$

Linear systems of ODEs

A system of two linear (homogeneous) differential equations is

$$
\begin{aligned}
x_1' &= a(t)x_1 + b(t)x_2 \\
x_2' &= c(t)x_1 + d(t)x_2
\end{aligned}
$$

where $x_1(t), x_2(t)$ are unknown functions, $a(t), b(t), c(t), d(t)$ are coefficients of the system.
Initial conditions: $x_1(0) = P, x_2(0) = Q$.

Example

Solve the system $x_1' = x_2, x_2' = -x_1$ with initial conditions $x_1(0) = 1, x_2(0) = 0$.
Solution: $x_1(t) = \cos t, x_2(t) = -\sin t$.

Linear algebra: Linear operators and matrices

Definition

A linear operator is a map $A\colon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$.

The matrix of this linear operator is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Definition (Multiplying matrices by vectors)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$$

Example

Compute $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(A) $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$; (B) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; (C) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$; (D) $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$;

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Linear systems

- Linear system has the form $Av = w$:
- A is a given matrix, w is a given vector, v is an unknown vector.
- In coordinates: $$\begin{aligned} ax_1 + bx_2 &= M \\ cx_1 + dx_2 &= N \end{aligned}$$

Example

Solve the linear system $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

(A) $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ (B) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (C) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (D) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

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Existence of solutions for linear systems

Definition

The *determinant* of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\boxed{\det A = ad - bc}$

Definition

A 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is *degenerate* if $\det A = 0$.

Theorem

If a 2×2 matrix A is non-degenerate, then the linear system $Av = w$ always has a unique solution. If the matrix is degenerate, the system will have no solutions or infinitely many solutions.

Example

Which of the following matrices are degenerate?

(A) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; (B) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; (C) $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$; (D) $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$

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Eigenvalues and eigenvectors

Definition

A number λ and a vector $v \neq 0$ are called an **eigenvalue** and the corresponding **eigenvector** of the matrix A if $\boxed{Av = \lambda v}$.

Example

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, thus $\lambda = 3$ and $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are an eigenvalue and an eigenvector of this matrix.

Theorem

λ is an eigenvalue of A if and only if $A - \lambda I$ is degenerate: $\boxed{\det(A - \lambda I) = 0}$.

Here $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a unit matrix.

Proof.

$Av = \lambda v$ is equivalent to $(A - \lambda I)v = 0$. If $A - \lambda I$ is non-degenerate, this will have a unique solution $v = 0$. So $A - \lambda I$ must be degenerate. $\square$

Finding eigenvalues and eigenvectors

Find eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

- $\det(A - \lambda I) = 0 \Rightarrow 0 = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda) - 4 = -3 - 2\lambda + \lambda^2$.
- Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = -1$.
- Now solve the systems $(A - \lambda I)v = 0$ for each of them.
- $\lambda_1 = 3$: $\begin{pmatrix} 1-3 & 2 \\ 2 & 1-3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \Rightarrow -2u + 2v = 0$
- Solutions: $(u, v) = (c, c)$ for any c. We can pick one of them, $(u, v) = (1, 1)$; others are proportional.

Example

Find the eigenvector that corresponds to $\lambda_2 = -1$.

(A) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; (B) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; (C) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$; (D) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

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Solution of the linear ODE

Theorem

If the matrix A has two different real eigenvalues λ_1, λ_2 with eigenvectors v_1, v_2, then solutions of the ODE $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *are*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_1 t} v_2$$

Example

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned} \qquad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}} :$$

$$x_1(t) = c_1 e^{3t} + c_2 e^{-t}, \quad x_2(t) = c_1 e^{3t} - c_2 e^{-t}$$