MATH 308. Differential Equations

Lecture 2. Direction fields. Euler's method

Nataliya Goncharuk

Texas A & M

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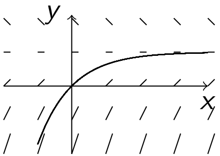
Direction fields for $y' = f(x, y)$: reminder

Solution curve Graph of solution;

Direction field Direction with slope $f(x, y)$ at each point (x, y).

Solution curves are tangent to all directions of the direction field

Reason: The tangent line to the solution curve has a slope $y'(x)$. But $y' = f(x, y)$ — so the slope is same as in the direction field!



Direction field for $y' = 1 - y$, with one solution curve that corresponds to the initial condition $y(0) = 0$.
Conclusion: $\lim_{x \to +\infty} y(x) = 1$.

Logistic equation: qualitative studies

- $y' = ky(c - y)$ — equation of population growth, limited resources. y is in thousands.

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Example

What is the limit population $\lim_{x \to +\infty} y(x)$ if the initial population is $y(0) = 0.5$?

(A) 0; (B) 1; (C) $+\infty$; (D) I don't know.

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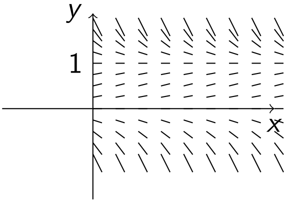

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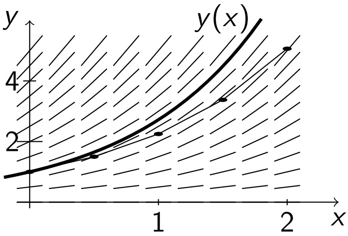
Computer-generated direction fields

Demonstrating geogebra.org/m/xzg6qtam, thanks to Prof. Rick Lynch.

Euler's method for solving ODEs numerically

- **Slope field:** a line with slope $f(x, y)$ at each point (x, y).



Euler's method for solving ODEs numerically

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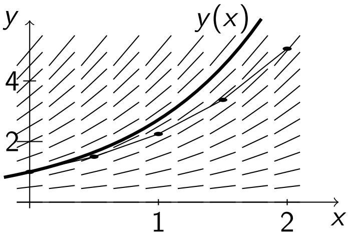
- **Solution curve** is tangent to all these lines, $y'(x) = f(x, y)$.

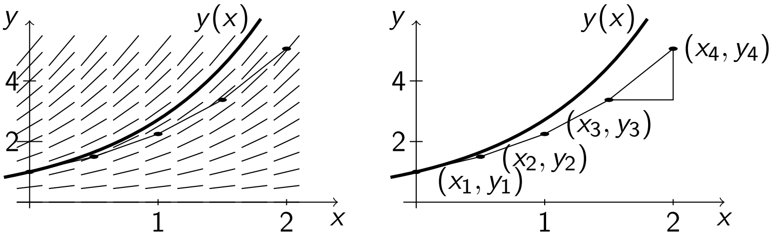


Euler's method for solving ODEs numerically

- **Slope field:** a line with slope $f(x, y)$ at each point (x, y).
- **Solution curve** is tangent to all these lines, $y'(x) = f(x, y)$.
- **Approximate solution curve** is a polygonal chain tangent to some lines, $\Delta y = \Delta x \cdot f(x, y)$ i.e.

$$y_{n+1} - y_n = s \cdot f(x_n, y_n),$$

where $t_{n+1} - t_n = s$ — step size of the method.



Example

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x_n	0	0.1	0.2	0.3	...
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x_n	0	0.1	0.2	0.3	...
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Using a Python program

Use a Python program implementing the Euler's method (e.g., this one) to find an approximate value of $y(5)$ for the ODE $y' = x - y$, with initial condition $y(0) = 2$.

(A) 0; (B) 2; (C) 4; (D) ∞.

Remark. The Euler's method is not the best numerical method even among simple ones. See e.g. Sec. 8.3 for the Runge-Kutta methods.