Lecture 20: Linear equations (Sec. 7.4)

MATH 308. Differential Equations

Nataliya Goncharuk

Texas A & M

November 8, 2022

Superposition principle; Existence and Uniqueness theorem

Theorem (Linearity of linear operators)

For any two vectors v_1, v_2, and any two scalars c_1, c_2, for a linear operator A, we have $A(c_1 v_1 + c_2 v_2) = c_1 A v_1 + c_2 A v_2$.

Theorem (Superposition principle)

If ϕ_1 and ϕ_2 are solutions of the linear equation $x' = Ax$, then $c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution of this equation.

Proof.

Indeed, $(c_1\phi_1 + c_2\phi_2)' = c_1\phi_1' + c_2\phi_2' = c_1 A\phi_1 + c_2 A\phi_2 = A(c_1\phi_1 + c_2\phi_2)$. □

Remark: here $\phi_1 = \phi_1(t), \phi_2 = \phi_2(t)$ are vectors that depend on t.

Theorem (Existence and Uniqueness theorem)

The solution of the initial value problem

$$x' = Ax, \qquad x_1(0) = a, x_2(0) = b$$

exists, is defined on $\mathbb{R}$, and is unique.

Solving linear equations using eigenvalues and eigenvectors

Theorem

If the matrix A has two different real eigenvalues λ_1, λ_2 with eigenvectors v_1, v_2, then the solutions of the ODE $x' = Ax$ are

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_1 t} v_2$$

Proof.

- Check that $e^{\lambda_1 t} v_1$ is a solution:
- $(e^{\lambda_1 t} v_1)' = \lambda e^{\lambda_1 t} v_1 \qquad A(e^{\lambda_1 t} v_1) = e^{\lambda_1 t} A v_1$
- These expressions are equal since $Av_1 = \lambda_1 v_1$.
- Same for $e^{\lambda_2 t} v_2$.
- Superposition principle $\rightarrow c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_1 t} v_2$ is a solution.
- We will see that we can satisfy all initial conditions, thus we have found all solutions (Existence and Uniqueness theorem).

□

How to satisfy initial conditions

Solve $x' = Ax, \quad x(0) = \begin{pmatrix} a \\ b \end{pmatrix}$

- $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$
- $x(0) = c_1 v_1 + c_2 v_2 = \begin{pmatrix} a \\ b \end{pmatrix}$
- Solve for c_1, c_2 (solve a linear system).
- Solution exists since v_1, v_2 are not proportional.
- This proves that we have found all solutions: we can satisfy all initial conditions.

Example

$$x' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

- $x(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- $x(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$
- $c_1 = c_2 = 1.$

Example

Solve

$$x_1' = 2x_1 + x_2$$
$$x_2' = -3x_2$$

- Write out the matrix A of the system.
- Find its eigenvalues.

(A) $1, 3$;

(B) $2, -3$;

(C) $0, 3$;

(D) $1, 2$.

Reminder: eigenvalues λ_1, λ_2 are two solutions of the equation $\det(A - \lambda I) = 0$.

https://pingo.coactum.de/885803

Finding eigenvectors

Solve

$$x_1' = 2x_1 + x_2$$
$$x_2' = -3x_2$$

Find an eigenvector corresponding to $\lambda = 2$.

(A) $(1, 0)$;

(B) $(0, 1)$;

(C) $(0, 0)$;

(D) $(1, -5)$.

Reminder: you should solve $(A - \lambda I)w = 0$.

https://pingo.coactum.de/885803

- The eigenvector corresponding to -3 equals $(1, -5)$.
- Solution is $x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -5 \end{pmatrix}$.

Initial conditions

Solve

$$x_1' = 2x_1 + x_2$$
$$x_2' = -3x_2$$

with initial conditions $x_1(0) = 0, x_2(0) = 5$.

- General solution: $x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -5 \end{pmatrix}$.
- We need to find c_1, c_2 to satisfy initial conditions.

(A) $c_1 = 0, c_2 = 5$;

(B) $c_1 = 5, c_2 = -5$;

(C) $c_1 = 1, c_2 = -1$;

(D) $c_1 = 0, c_2 = -1$.

https://pingo.coactum.de/885803

Complications

Theorem

If the matrix A has two different real eigenvalues λ_1, λ_2 with eigenvectors v_1, v_2, then solutions of the ODE $x' = Ax$ are

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_1 t} v_2$$

Reminder: eigenvalues λ_1, λ_2 are solutions of the quadratic equation $\det(A - \lambda I) = 0$.

What can go wrong?

- Complex eigenvalues;
- Repeated eigenvalues.

Sympy for eigenvalues and eigenvectors; phase curves.

[Finding eigenvalues and eigenvectors using sympy functions .eigenvals() and .eigenvects().]

- If $(x_1(t), x_2(t))$ is the solution of the system $x' = Ax$, then any curve $(x_1(t), x_2(t))$ on the Ox_1x_2 plane is called the **phase curve.**
- Using wolframalpha's Streamplot to plot phase curves.
- **Example:** the phase curve $(e^{-3t}, -5e^{-3t})$ for the previous equation is the ray $x_2 = -5x_1$ along the eigenvector $(1, -5)$.
- The phase curve $(e^{2t}, 0)$ for the previous equation is the ray $x_2 = 0$ along the eigenvector $(1, 0)$.
- In general, you will see four rays along both eigenvectors.