Lecture 22: Linear Systems of ODEs: complex and repeated eigenvalues

MATH 308. Differential Equations

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Complex eigenvectors

Example: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

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Example

Find an eigenvector of the matrix $A = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}$ corresponding to the eigenvalue $-1 + \sqrt{2}i$.

(A) $\begin{pmatrix} 1 - \sqrt{2}i \\ 3 \end{pmatrix}$ (B) $\begin{pmatrix} 1 \\ \sqrt{2}i - 1 \end{pmatrix}$ (C) $\begin{pmatrix} 1 + \sqrt{2}i \\ -3 \end{pmatrix}$ (D) IDK

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Eigenvalue method for complex eigenvalues

Theorem

If the 2×2 matrix A has 2 complex eigenvalues $\lambda_1, \lambda_2 = a \pm ib$ with eigenvectors $v_{1,2}$, then the solutions of the ODE $x' = Ax$ are

$$x(t) = c_1 \mathrm{Re}\left(e^{\lambda_1 t} v_1\right) + c_2 \mathrm{Im}\left(e^{\lambda_1 t} v_1\right)$$

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- $a = -1, b = \sqrt{2}$, $k = (1, -1)$, $l = (0, \sqrt{2})$.
- $x(t) = c_1 e^{-t}(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\sqrt{2}t - \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} \sin\sqrt{2}t) +$
  $c_2 e^{-t}(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin\sqrt{2}t + \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} \cos\sqrt{2}t)$

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- $x(t) = c_1 e^{-t}(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos \sqrt{2}t - \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} \sin \sqrt{2}t) +$
  $c_2 e^{-t}(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin \sqrt{2}t + \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} \cos \sqrt{2}t)$
- In particular, $x_1(t) = c_1 e^{-t}\cos\sqrt{2}t + c_2 e^{-t}\sin\sqrt{2}t$ (cf. last time).

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A matrix A has an eigenvalue $2 + 3i$ with an eigenvector $(1 + 2i, i)$. Find $x_1(t)$ using the above formula.

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A matrix A has an eigenvalue $2 + 3i$ with an eigenvector $(1 + 2i, i)$. Find $x_1(t)$ using the above formula.

(A) $c_1 e^{2t} \cos 3t + c_2 e^{2t} \sin 3t$

(B) $c_1 e^{2t}(\cos 3t - 2 \sin 3t) + c_2 e^{2t}(\sin 3t + 2 \cos 3t)$

(C) $c_1 e^{2t}(\cos 3t - \sin 3t) + c_2 e^{2t}(\sin 3t + \cos 3t)$

(D) $c_1 e^{2t}(2 \cos 3t - \sin 3t) + c_2 e^{2t}(2 \sin 3t + \cos 3t)$

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- If A has a repeated eigenvalue λ with the only eigenvector v, then

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- We get
$$A\xi = \lambda\xi + \eta, A\eta = \lambda\eta,$$
- η is the eigenvector; take $\eta = v$. ξ is a *generalized eigenvector:* $(A - \lambda I)\xi = v$.

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Any non-eigenvector can serve as ξ.

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$$A = \begin{pmatrix} 5 & 4 \\ -1 & 1 \end{pmatrix} \qquad \lambda^2 - 6\lambda + 9 = 0 \qquad \lambda_{1,2} = 3$$

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Find the only eigenvalue of $A = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}$; using $\xi = (1, 0)$ as the generalized eigenvector, find the corresponding v.

(A) $v = (0, -4)$;
(B) $v = (-2, -2)$;
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(D) $v = (-2, -6)$.

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- Generalizations to higher dimension require the Jordan basis.