Lecture 23: Fundamental matrices (Sec. 7.7)

MATH 308. Differential Equations

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Fundamental matrices

Solutions $x(t)$ and $\tilde{x}(t)$ of the 2×2-system $x' = Ax$ are called dependent if they are proportional: $\tilde{x} = cx$ or $x = c\tilde{x}$ where c is a constant.

Definition

A fundamental matrix $\Psi(t)$ for a system $x' = Ax$ is a matrix whose columns are independent solutions.

Example

Which of the following are fundamental matrices for the system $x_1' = x_2, x_2' = -x_1$? [Multiple choice]

(A) $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

(B) $\begin{pmatrix} \cos t & 0 \\ -\sin t & 0 \end{pmatrix}$

(C) $\begin{pmatrix} \cos t + \sin t & \cos t \\ -\sin t + \cos t & -\sin t \end{pmatrix}$

(D) $\begin{pmatrix} \cos t - \sin t & \sin t - \cos t \\ -\sin t + \cos t & \cos t + \sin t \end{pmatrix}$

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Relation to the general formula for solutions of $x' = Ax$

Example

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- Solutions:

$$x(t) = c_1 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

- $\Psi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}$.
- Note that:

$$x(t) = c_1 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Psi(t)c$$

- $\boxed{x(t) = \Psi(t)c}$

Matrix exponential e^{At}: best fundamental matrix ever

Definition

The matrix exponential e^{At} is the fundamental matrix whose columns are solutions with initial conditions $(1,0)$ and $(0,1)$: $e^{A\cdot 0} = I$.

Example

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- $\begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}$ is a fundamental matrix, but not the matrix exponential.
- Solutions with initial conditions $(1,0)$ and $(0,1)$ are $x(t) = \begin{pmatrix} \frac{e^t+e^{-t}}{2} \\ \frac{e^t-e^{-t}}{2} \end{pmatrix}$ and $x(t) = \begin{pmatrix} \frac{e^t-e^{-t}}{2} \\ \frac{e^t+e^{-t}}{2} \end{pmatrix}$
- $e^{At} = \begin{pmatrix} \frac{e^t+e^{-t}}{2} & \frac{e^t-e^{-t}}{2} \\ \frac{e^t-e^{-t}}{2} & \frac{e^t+e^{-t}}{2} \end{pmatrix}$.

Example

Definition

The matrix exponential e^{At} is the fundamental matrix whose columns are solutions with initial conditions $(1, 0)$ and $(0, 1)$: $e^{A\cdot 0} = I$.

Example

Find $e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t}$.

(A) $\begin{pmatrix} 1 & e^t \\ 1 & 1 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}$

(C) $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

(D) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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Using e^{At} to solve $x' = Ax$

Theorem

Solution of $x' = Ax$, $x(0) = x_0$ is given by $\boxed{x(t) = e^{At}x_0}$.

Proof.

As for any other fundamental matrix, we have $x = \Psi(t)c$, i.e.

$$x(t) = e^{At}c.$$

But now $x_0 = x(0) = e^{A\cdot 0}c = c$, i.e. c coincides with initial condition. □

This makes e^{At} the best fundamental matrix ever: we do not need to find c using initial conditions!

Example

Theorem

Solution of $x' = Ax$, $x(0) = x_0$ *is given by* $\boxed{x(t) = e^{At}x_0}$.

Example

Given that $e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, solve the initial value problem $x' = Ax$, $x(0) = (1, 3)$.

Solution:

$$x(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \cos t + 3 \sin t \\ -\sin t + 3 \cos t \end{pmatrix}$$

Meaning of the notation e^{At}; alternative definition

Definition

Matrices are multiplied row-by-column:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} f & g \\ h & k \end{pmatrix} = \begin{pmatrix} af + bh & ag + bk \\ cf + dh & cg + dk \end{pmatrix}$$

Definition

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \dots$$

where $B^2 = B \cdot B, B^3 = B \cdot B \cdot B \dots$

Definition

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots$$

(this way of computing is NOT numerically effective).

Example

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

Example

$$e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} + \dots =$$

$$\begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Example: Compute $e^{\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} t}$.

(A) $\begin{pmatrix} 1 & 1 \\ e^{2t} & 1 \end{pmatrix}$

(B) $\begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 0 \\ 2t & 1 \end{pmatrix}$

(D) $\begin{pmatrix} 1 + \frac{t^2}{2} + \frac{t^4}{4!} \dots & 0 \\ 2t + \frac{2t^3}{3!} + \dots & 1 + \frac{t^2}{2} + \frac{t^4}{4!} \dots \end{pmatrix}$

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