Lecture 25: Nonhomogeneous systems (Sec. 7.9)

MATH 308. Differential Equations

Nataliya Goncharuk

Texas A & M

December 1, 2022

Reminder: matrix exponentials

Compute $\exp(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t)$ using the fact that the columns are solutions of $x' = Ax$ with initial conditions $x(0) = (1, 0)$ and $x(0) = (0, 1)$.

Review: Linear ODEs

$$y' + p(t)y = q(t)$$

- Integrating multiplier: $Z = e^{\int p(t)dt}$

Review: Linear ODEs

$$y' + p(t)y = q(t)$$

- Integrating multiplier: $Z = e^{\int p(t)dt}$
- $$y' e^{\int p(t)dt} + p(t)e^{\int p(t)dt} y = e^{\int p(t)dt} q(t)$$

Review: Linear ODEs

$$y' + p(t)y = q(t)$$

- Integrating multiplier: $Z = e^{\int p(t)dt}$
- $$y'e^{\int p(t)dt} + p(t)e^{\int p(t)dt}y = e^{\int p(t)dt}q(t)$$
- $$(ye^{\int p(t)dt})' = e^{\int p(t)dt}q(t)$$

Review: Linear ODEs

$$y' + p(t)y = q(t)$$

- Integrating multiplier: $Z = e^{\int p(t)dt}$
- $$y'e^{\int p(t)dt} + p(t)e^{\int p(t)dt}y = e^{\int p(t)dt}q(t)$$
- $$(ye^{\int p(t)dt})' = e^{\int p(t)dt}q(t)$$
- $$ye^{\int p(t)dt} = \int e^{\int p(t)dt}q(t)dt$$

Review: Linear ODEs

$$y' + p(t)y = q(t)$$

- Integrating multiplier: $Z = e^{\int p(t)dt}$
- $$y'e^{\int p(t)dt} + p(t)e^{\int p(t)dt}y = e^{\int p(t)dt}q(t)$$
- $$(ye^{\int p(t)dt})' = e^{\int p(t)dt}q(t)$$
- $$ye^{\int p(t)dt} = \int e^{\int p(t)dt}q(t)dt$$
- $$y = e^{-\int p(t)dt}\int e^{\int p(t)dt}q(t)dt$$

Case $x' = ax + q(t)$ and its generalization

- $p(t) = -a$, $\int -p(t)dt = at$

Case $x' = ax + q(t)$ and its generalization

- $p(t) = -a$, $\int -p(t)dt = at$
- $$y = e^{-\int p(t)dt} \int e^{\int p(t)dt} q(t)dt = e^{at} \int e^{-at} q(t)dt$$

Case $x' = ax + q(t)$ and its generalization

- $p(t) = -a$, $\int -p(t)dt = at$
- $$y = e^{-\int p(t)dt} \int e^{\int p(t)dt} q(t)dt = e^{at} \int e^{-at} q(t)dt$$
- This generalizes to $x' = Ax + b(t)$:

Case $x' = ax + q(t)$ and its generalization

- $p(t) = -a$, $\int -p(t)dt = at$
- $$y = e^{-\int p(t)dt} \int e^{\int p(t)dt} q(t)dt = e^{at} \int e^{-at} q(t)dt$$
- This generalizes to $x' = Ax + b(t)$:

Proposition (Variation of Parameters)

Solutions of $x' = Ax + b(t)$ are given by

$$\boxed{x(t) = e^{At} \int e^{-At} b(t)dt}$$

How to use this formula

$$\boxed{x(t) = e^{At} \int e^{-At} b(t) dt}$$

- e^{At} is the matrix exponential.

How to use this formula

$$\boxed{x(t) = e^{At} \int e^{-At} b(t) dt}$$

- e^{At} is the matrix exponential.
- $e^{-At} = e^{A(-t)}$: same matrix with $-t$ instead of t.

How to use this formula

$$\boxed{x(t) = e^{At} \int e^{-At} b(t) dt}$$

- e^{At} is the matrix exponential.
- $e^{-At} = e^{A(-t)}$: same matrix with $-t$ instead of t.
- $e^{-At} b(t)$ is a (row-by-column) product of a matrix and a vector.

How to use this formula

$$\boxed{x(t) = e^{At} \int e^{-At} b(t) dt}$$

- e^{At} is the matrix exponential.
- $e^{-At} = e^{A(-t)}$: same matrix with $-t$ instead of t.
- $e^{-At} b(t)$ is a (row-by-column) product of a matrix and a vector.
- $\int e^{-At} b(t)$: integrate each term of the vector.

Example

$$\boxed{x(t) = e^{At} \int e^{-At} b(t) dt}$$

$x_1' = x_2, x_2' = -x_1 + \frac{1}{\sin t}$

Example

$$\boxed{x(t) = e^{At} \int e^{-At} b(t) dt}$$

Why does this formula hold true

- Solution of $x' = Ax$ is $x(t) = e^{At}c$.

Why does this formula hold true

- Solution of $x' = Ax$ is $x(t) = e^{At}c$.
- Since $e^{At}c$ solves $x' = Ax$, we have $(e^{At}c)' = Ae^{At}c$. So

Why does this formula hold true

- Solution of $x' = Ax$ is $x(t) = e^{At}c$.
- Since $e^{At}c$ solves $x' = Ax$, we have $(e^{At}c)' = Ae^{At}c$. So
- $(e^{At})'c = Ae^{At}c$ for any c, i.e. $\boxed{(e^{At})' = Ae^{At}}$.

Why does this formula hold true

- Solution of $x' = Ax$ is $x(t) = e^{At}c$.
- Since $e^{At}c$ solves $x' = Ax$, we have $(e^{At}c)' = Ae^{At}c$. So
- $(e^{At})'c = Ae^{At}c$ for any c, i.e. $\boxed{(e^{At})' = Ae^{At}}$.
- Variate parameter c: $x(t) := e^{At}c(t)$ will solve $x' = Ax + b(t)$.

Why does this formula hold true

- Solution of $x' = Ax$ is $x(t) = e^{At}c$.
- Since $e^{At}c$ solves $x' = Ax$, we have $(e^{At}c)' = Ae^{At}c$. So
- $(e^{At})'c = Ae^{At}c$ for any c, i.e. $\boxed{(e^{At})' = Ae^{At}}$.

- Variate parameter c: $x(t) := e^{At}c(t)$ will solve $x' = Ax + b(t)$.
- LHS $x' = (e^{At}c(t))' = Ae^{At}c(t) + e^{At}c'(t)$ equals RHS $Ax + b(t) = Ae^{At}c(t) + b(t)$

Why does this formula hold true

- Solution of $x' = Ax$ is $x(t) = e^{At}c$.
- Since $e^{At}c$ solves $x' = Ax$, we have $(e^{At}c)' = Ae^{At}c$. So
- $(e^{At})'c = Ae^{At}c$ for any c, i.e. $\boxed{(e^{At})' = Ae^{At}}$.

- Variate parameter c: $x(t) := e^{At}c(t)$ will solve $x' = Ax + b(t)$.
- LHS $x' = (e^{At}c(t))' = Ae^{At}c(t) + e^{At}c'(t)$ equals RHS $Ax + b(t) = Ae^{At}c(t) + b(t)$
- We get $e^{At}c'(t) = b(t)$.

Why does this formula hold true

- Solution of $x' = Ax$ is $x(t) = e^{At}c$.
- Since $e^{At}c$ solves $x' = Ax$, we have $(e^{At}c)' = Ae^{At}c$. So
- $(e^{At})'c = Ae^{At}c$ for any c, i.e. $\boxed{(e^{At})' = Ae^{At}}$.

- Variate parameter c: $x(t) := e^{At}c(t)$ will solve $x' = Ax + b(t)$.
- LHS $x' = (e^{At}c(t))' = Ae^{At}c(t) + e^{At}c'(t)$ equals RHS $Ax + b(t) = Ae^{At}c(t) + b(t)$
- We get $e^{At}c'(t) = b(t)$.

- **Lemma**: $e^{-At}e^{At} = I$.

Why does this formula hold true

- Solution of $x' = Ax$ is $x(t) = e^{At}c$.
- Since $e^{At}c$ solves $x' = Ax$, we have $(e^{At}c)' = Ae^{At}c$. So
- $(e^{At})'c = Ae^{At}c$ for any c, i.e. $\boxed{(e^{At})' = Ae^{At}}$.

- Variate parameter c: $x(t) := e^{At}c(t)$ will solve $x' = Ax + b(t)$.
- LHS $x' = (e^{At}c(t))' = Ae^{At}c(t) + e^{At}c'(t)$ equals RHS $Ax + b(t) = Ae^{At}c(t) + b(t)$
- We get $e^{At}c'(t) = b(t)$.

- **Lemma**: $e^{-At}e^{At} = I$.
- $e^{-At}e^{At}c'(t) = c'(t) = e^{-At}b(t)$

Why does this formula hold true

- Solution of $x' = Ax$ is $x(t) = e^{At}c$.
- Since $e^{At}c$ solves $x' = Ax$, we have $(e^{At}c)' = Ae^{At}c$. So
- $(e^{At})'c = Ae^{At}c$ for any c, i.e. $\boxed{(e^{At})' = Ae^{At}}$.

- Variate parameter c: $x(t) := e^{At}c(t)$ will solve $x' = Ax + b(t)$.
- LHS $x' = (e^{At}c(t))' = Ae^{At}c(t) + e^{At}c'(t)$ equals RHS $Ax + b(t) = Ae^{At}c(t) + b(t)$
- We get $e^{At}c'(t) = b(t)$.

- **Lemma**: $e^{-At}e^{At} = I$.
- $e^{-At}e^{At}c'(t) = c'(t) = e^{-At}b(t)$
- $c(t) = \int e^{-At}b(t)dt$ q.e.d.

Why $e^{-At}e^{At} = I$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots$$

Corollary

We have $e^{A(-t)}e^{At} = I$.

Why $e^{-At}e^{At} = I$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots$$

Corollary

We have $e^{A(-t)}e^{At} = I$.

Proof.

Open all brackets in $(I - At + \frac{A^2t^2}{2!} - \dots)(I + At + \frac{A^2t^2}{2!} + \dots)$ — everything except I cancels out. $\square$

Why $e^{-At}e^{At} = I$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots$$

Corollary

We have $e^{A(-t)}e^{At} = I$.

Proof.

Open all brackets in $(I - At + \frac{A^2t^2}{2!} - \dots)(I + At + \frac{A^2t^2}{2!} + \dots)$ — everything except I cancels out. □

Warning: $e^{A+B} \neq e^A e^B$!!!

Only works if $AB = BA$.