MATH 308. Differential Equations

Lecture 3. Separable Differential Equations

Nataliya Goncharuk

Texas A & M

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Ordinary vs Partial Differential Equations (Sec 1.3)

Ordinary DEs relate derivatives of a function with respect to a single variable. Examples:

$$\frac{d^2 r}{dt^2} = -\frac{\gamma M}{r^2};$$

$$y''' = -y.$$

Partial DEs relate derivatives of a function of multiple variables with respect to different variables. Examples:

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = k^2 \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right);$$

$$z'_t = z''_{xx}$$

Equations vs Systems (Sec 1.3)

Differential Equations deal with a single unknown function.

Example:

$$\frac{d^2 r}{dt^2} = -\frac{\gamma M}{r^2};$$

Systems of DEs deal with multiple unknown functions and have many equations; example:

$$\frac{dx}{dt} = y;$$
$$\frac{dy}{dt} = -x.$$

Order of a Differential Equation (Sec 1.3)

Definition

The *order* of a differential equation is the highest derivative that appears in the equation.

Equation	Order
$y''(x) = 2y'(x) + y^3(x)$	2
$\frac{d^2x(t)}{dt^2} = \frac{\gamma M}{x(t)^2}$	2
$\frac{\partial^2 u(x,y,t)}{\partial t^2} = k^2 \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right)$	2
$y''(x)y(x) + \sin(y'''(x))y'(x) = 1$	3

What are the orders of these differential equations?

(A) $2, 2, 2, 3$

(B) $3, 2, 2, 3$

(C) $2, 3, 3, 3$

(D) I don't know.

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Separable Differential Equations (Sec. 2.2)

Definition

A *separable* differential equation is a *first order* differential equation of the form

$$\frac{dy}{dx} = f(x)h(y).$$

Can be written in a different form, e.g. $y' = \frac{f(x)}{g(y)}$ or

$$M(y)y' + N(x) = 0.$$

Which of the following equations are separable? Select all.

(A) $y'(x) = y(x) + x$;

(B) $(\sin x)y' = y^2$;

(C) $y''(x) = y(x)$;

(D) $e^{\sin x} + e^{y(x)}y'(x) = 5$?

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Separable Differential Equations (Sec. 2.2): solving

$$y' = \frac{f(x)}{g(y)}$$

Notation: $F' = f$, $G' = g$.
Rewrite the equation as

$$g(y)y' = f(x).$$

The right-hand side is the derivative of $F(x)$.
The left-hand side is the derivative of $G(y(x))$ due to the Chain Rule:

$$\frac{d}{dx}G(y) = G'(y) \cdot y' = g(y) \cdot y'.$$

Thus we can integrate both sides:

$$G(y) = F(x) + c$$

(implicit formula for the function $y(x)$).

Way to use the method

Equation:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Separate variables:

$$g(y)dy = f(x)dx$$

Integrate:

$$\int g(y)dy = \int f(x)dx$$

Implicit solution:

$$G(y) = F(x) + c.$$

Example

Equation $\frac{dy}{dx} = \frac{-x}{y}$, $\quad y(0) = -1.$

Separate variables: $y\,dy = -x\,dx.$

Integrate: $\dfrac{y^2}{2} = \dfrac{-x^2}{2} + C,$

Solve for y: $y(x) = \pm\sqrt{-x^2 + c}.$

Find c and the sign: $y(0) = \pm\sqrt{-0 + c} = -1$, thus
$c = 1, y(x) = -\sqrt{-x^2 + 1}.$

Example

Solve the following equation: $y' = \frac{\sin x + 1}{y+1}$, $y(0) = 0$. You may leave your answer in the implicit form.

(A) $\frac{y^2}{2} + y = \sin x;$

(B) $\frac{y^2}{2} + y = -\cos x + x + 1;$

(C) $y = (-\cos x + x + 1)\ln|y + 1|$

(D) I don't know

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Domains of solutions

[Showing the desmos.com graph for $\frac{y^2}{2} + y = -\cos x + x + 1$]

Example

What is the domain of the function y?

(A) $\mathbb{R}$;

(B) $(-0.8, +\infty)$;

(C) $(-2, 0)$;

(D) I don't know.

Comment: Domain of the solution of the ODE is always connected: if it has several pieces, these are considered as several separate solutions.

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Separable Differential Equations (Sec. 2.2)

Constant solutions

- $\frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx \Rightarrow$
- $\int \frac{dy}{y} = \int dx$
- $\ln |y(x)| = x + C$
- $|y(x)| = e^C e^x$
- $y(x) = \pm e^C e^x$
- $y(x) = \pm e^C e^x$.

But $y = 0$ is also a solution! How did we lose it?

Division by zero

- If $y(x) = 0$, then we may not divide both sides by y.
- So, the correct answer is $y(x) = \pm e^C e^x$ or $y(x) = 0$.
- Can be written as $y(x) = ce^x$.

Constant solutions: general case

$$y' = f(x)h(y)$$

If $h(c) = 0$, then $y(x) = c$ is a solution:

$$c' = 0 = f(x)h(c) = 0$$

and it will be lost when you divide by $h(y)$.

Example

Which solutions should be added after you solve $y' = \frac{y^2-4}{y}$ using separation of variables? Mark all correct options.

(A) 2; (B) -2; (C) 4; (D) 0.

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Computer demo

[Time permitting, showing how the Python program https://mybinder.org/v2/gh/urkud/ODE-notebooks/master?filepath=Euler's method.ipynb) behaves for $y' = -x/y, y(0) = 1$, x running from 0 to 1.3.]