MATH 308. Differential Equations

Lecture 4. First-order linear differential equations

Nataliya Goncharuk

Texas A & M

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Linear vs nonlinear DEs (Sec. 1.3)

Definition

A *linear differential equation* is an equation of the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t),$$

possibly with some terms moved around between LHS and RHS.
All other differential equations are called *nonlinear*.

Which of the following differential equations are linear? Select all.

(A) $x^2 y''(x) + (\sin x)y(x) = e^x$; (B) $e^{xy'(x)} = 1$;

(C) $y'(x) = x \sin y(x)$; (D) $y'(x) = y(x) \sin x$.

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Chain rule

Find derivative of $e^{\frac{x^2}{2}}y$, where $y = y(x)$, with respect to x

(A) $xye^{\frac{x^2}{2}}$;

(B) $xye^{\frac{x^2}{2}} + y'e^{\frac{x^2}{2}}$;

(C) $y'e^{\frac{x^2}{2}}$;

(D) I don't know.

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First order linear differential equations

Example

$$y' + xy = x.$$

Multiply both sides by $e^{\frac{x^2}{2}}$ $e^{\frac{x^2}{2}} y' + e^{\frac{x^2}{2}} xy = e^{\frac{x^2}{2}} x;$

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Evaluate the integral $\quad e^{\frac{x^2}{2}} y = e^{\frac{x^2}{2}} + C;$

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How did I guess the integrating factor $e^{\frac{x^2}{2}}$?

First order linear differential equations

Generic equation

- $y' + p(x)y = g(x).$

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First order linear differential equations

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- $y' + p(x)y = g(x)$.
- Multiply both sides by $\mu(x)$: $\mu(x)y' + \mu(x)p(x)y = \mu(x)g(x)$

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$$(\mu(x)y)' = \mu(x)y' + \mu'(x)y.$$

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Example

Separate variables and find its solution.

(A) $\mu(x) = ce^{p(x)}$; (B) $\mu(x) = ce^{\int p(x)dx}$;

(C) $\mu(x) = \frac{p(x)^2}{2} + c$; (D) I don't know.

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- Integrate both sides, divide by μ

$$y = \frac{1}{\mu(x)} \int \mu(x)g(x)\, dx$$

First order linear differential equations, Example

Equation $y' + y/x = 1$.

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First order linear differential equations, Example

Equation $y' + y/x = 1.$

Integrating factor $\mu(x) = \exp \int 1/x dx = \exp \ln |x| = |x|.$

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Use $\mu(x) = x$.

We get $xy' + y = x$ or $(xy)' = x$.

Integrate $xy = x^2/2 + c$;

Answer $y = x/2 + c/x$.

Example

Solve the equation $y' - 2xy = e^{x^2}$.

(A) $x + c$;

(B) $1 + ce^{-x^2}$;

(C) $(x + c) \cdot e^{x^2}$;

(D) I don't know.

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Behavior of solutions of linear equations

Remark: solutions have the form

$$y(x) = A(x) + c\frac{1}{\mu(x)}.$$

If $\frac{1}{\mu(x)} \to 0$ as $x \to +\infty$, different solutions approach each other.

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$$y' = k(T_{outside} - y)),$$

coefficient k depends on insulation.

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